Thermal Transport in a Luttinger Liquid

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We study thermal transport in a one-dimensional (1D) interacting electron gas, employing the Luttinger liquid model. Both thermal conductance and thermopower are analyzed for a pure 1D gas and with impurities. The universal ratio of electrical to thermal conductance in a Fermi liquid—the Wiedemann-Franz law—is modified, whereas the thermopower is still linear in temperature. For a single impurity the Lorentz number is given by $L(T \to 0) = 3L_0/(2g + g^2)$—with L_0 the Fermi liquid value—and the conductance $1/2 < g < 1$. For $g < 1/2$ the Lorentz number *diverges* as $T \to 0$. Possible relevance to thermal transport in conducting polymer systems is discussed.

The Wiedemann-Franz law, which relates the thermal and electrical conductivity (κ, σ) of metals, played a central role in the historical development of the quantum theory of solids. The Lorentz number, $L = \kappa/\sigma T$, originally computed within classical Drude theory, gave fortuitous agreement with experiment due to canceling errors. The quantum theory corrected the errors and improved the agreement. For noninteracting electrons Chester and Thellung [1] showed that the Lorentz number is given exactly by $L_0 = (\pi^2/3)(k_B/e)^2$, for *arbitrary* impurity scattering strength. In the 1980s Castellani *et al.* [2] argued that this universal value was robust even with the inclusion of electron interactions, provided the system remained metallic. Thus, a universal Lorentz number appears to be a defining characteristic of the Fermi-liquid phase.

In recent years there has been tremendous interest in conducting phases which are *not* Fermi liquids. A paradigm for these are 1D interacting electron gas models, which exhibit a non-Fermi-liquid phase even for weak interactions [3]. The resulting Luttinger liquid phase is characterized by a dimensionless conductance, g, which controls various power laws, such as the singularity in the momentum distribution function. The resurgence of interest in the 1D Luttinger liquids stems both from the recent ability to lithographically pattern true one-channel quantum wires [4,5] and from the realization that 1D edge states in the fractional quantum Hall effect are Luttinger liquids [6]. Other non-Fermi-liquid phases arise in quantum impurity problems [7], such as the multichannel Kondo model which is possibly relevant to heavy fermion materials. Bulk 2D non-Fermi-liquid phases have also been suggested in compressible Hall fluid phases [8] and in the cuprate superconductors [9].

It is natural to anticipate that thermal transport in such non-Fermi-liquid phases will be qualitatively different, and might help characterize and distinguish them experimentally. In this paper, we consider in detail thermal transport in the 1D Luttinger liquids. We show that the Lorentz number can be substantially modified from its Fermi-liquid value, L_0. The thermopower, Q, on the other hand, shows characteristically metallic behavior, $Q = cT$. As in conventional metals, the coefficient c is nonuniversal, depending on the curvature of the energy bands and the energy dependence of the scattering rates.

While thermal transport measurements in quantum wires and quantum Hall samples are undoubtedly extremely challenging, a remarkable recent experiment has demonstrated the feasibility of such experiments [10]. Thermal transport measurements in bulk quasi-1D samples, such as conducting polymers, are much easier, but 3D crossovers may tend to complicate the analysis.

Pure Luttinger liquid.—We begin with a model for an interacting spinless 1D electron gas in the absence of any impurities, which has a bosonized Hamiltonian density

$$\mathcal{H}_0 = \pi v_0(N_+^2 + N_-^2 + 2\lambda N_+ N_-). \qquad (1)$$

The right and left moving electron densities, $N_\pm$, satisfy Kac-Moody commutation relations,

$$[N_\pm(x), N_\pm(x')] = \pm(i/2\pi)\partial_x \delta(x - x'). \qquad (2)$$

The interaction term mixes right and left movers, but can be shifted away as usual by defining new fields

$$N_\pm = [g(n_+ + n_-) \pm (n_+ - n_-)]/2g, \qquad (3)$$

with $\lambda = (1 - g^2)/(1 + g^2)$. In terms of $n_\pm$ the Hamiltonian decouples into right and left moving sectors,

$$\mathcal{H}_0 = \mathcal{H}_0^+ + \mathcal{H}_0^- = (\pi v/g)(n_+^2 + n_-^2), \qquad (4)$$

with renormalized velocity $v = 2g v_0/(1 + g^2)$. The new fields also satisfy a Kac-Moody algebra,

$$[n_\pm(x), n_\pm(x')] = \pm(ig/2\pi)\partial_x \delta(x - x'). \qquad (5)$$

Consider now transport in an ideal Luttinger liquid. Initially, we ignore additional anharmonic interaction terms

(e.g., $n_+^2 n_-$) which couple the right and left moving modes in (4). (For chiral quantum Hall edge states, these will be absent.) It then suffices to consider only a single right moving channel, $n = n_+$. Such an ideal chiral channel can be characterized by transport coefficients $\mathcal{L}^{ij}$, which relate changes in the electrical and thermal currents to changes in the chemical potential, μ, and temperature T. These coefficients are equivalent to Landauer two-terminal transport coefficients [11], defined with "ideal" reservoirs. For ideal quantum wires, these coefficients are not measured directly, since the contacts do not couple selectively to right and left moving modes. However, they can be measured directly for quantum Hall edge states.

The charge density in a chiral channel is conserved by (4), and satisfies $\partial_t n + \partial_x J = 0$ with an electrical current $J = vn$. Changing the chemical potential, μ, alters the electrical current. Balancing the n^2 energy in (4) with a $-\mu n$ term gives $\Delta J = (g/2\pi)\Delta\mu$ or upon restoring units an electrical conductance, $G = \mathcal{L}^{11} = ge^2/h$.

Heat carried by a chiral channel is likewise conserved by (4). The continuity equation $\partial_t n_Q + \partial_x J_Q = 0$ is satisfied by the thermal energy density $n_Q = (\pi v/g)n^2$ and thermal current $J_Q = v n_Q$. The thermal energy at temperature T can be expressed in terms of the chiral Luttinger modes as

$$n_Q = \int_0^\infty \frac{dk}{2\pi}\, \omega_k b_{\omega_k}, \qquad (6)$$

with $b_\omega = (e^{\beta\omega} - 1)^{-1}$ and $\omega_k = vk$. This gives $J_Q = (\pi^2/6)(k_B T)^2/h$, and leads to a "quantized" thermal conductance, $K = \mathcal{L}^{22} = \partial J_Q/\partial T = (\pi^2/3)k_B^2 T/h$.

For an ideal Luttinger liquid we can then define a "two-terminal" Lorentz number

$$L_{\text{ideal}} = K/TG = L_0/g. \qquad (7)$$

For $g = 1$, we recover the Fermi-liquid value $L_0 = (\pi^2/3)(k_B/e)^2$. With repulsive interactions ($g < 1$) the Lorentz number is larger.

The off-diagonal transport coefficient $\mathcal{L}^{12} = \partial J/\partial T$, which determines the thermopower, is zero within the present model, due to the implicit linearization of the electronic band structure near the Fermi energy. The effects of dispersion can be included via the third order interaction term $\mathcal{H}_{\text{int}} = An^3$, which is normally ignored because it is formally "irrelevant." The coefficient A is proportional to the change in Fermi velocity with chemical potential, $dv/d\mu$. The resulting thermopower, $Q = \mathcal{L}^{12}/\mathcal{L}^{11}$, is linear in temperature [12],

$$Q = -(\pi^2 k_B^2/3gev)(dv/d\mu)T. \qquad (8)$$

In quantum wires, anharmonic interactions ignored above will couple the right and left moving modes. The right and left moving thermal currents will no longer be independently conserved. However, in a translationally invariant system, thermal currents cannot fully relax due to constraints of momentum conservation. Such anhar-monic interactions might nevertheless effect the value of K. Umklapp processes would allow a decay of thermal current, but freeze out at low temperatures. In any event, impurity backscattering will dominate these interaction effects in the thermal resistance of real quantum wires.

Single impurity.— We now consider a single impurity in an otherwise ideal Luttinger liquid as a first step toward inclusion of many impurities. (A single impurity is also relevant to point contact experiments in the quantum Hall effect.) A weak potential scatterer at the origin can be modeled by adding a term to the Hamiltonian,

$$\mathcal{H}_{\text{back}} = -t_B \cos(\phi_+ - \phi_-)\delta(x), \qquad (9)$$

where t_B is the amplitude for $2k_F$ electron backscattering. This process has been expressed in terms of the boson fields $\phi_\pm$, related to the densities $n_\pm = \pm\partial_x\phi_\pm/2\pi$. The operator $\exp(i\phi_+)$ creates an excitation with fractional charge, ge. Thus each backscattering process reflects fractional electron charge.

An impurity which strongly backscatters can alternatively be modeled as a tunnel junction between two decoupled semi-infinite Luttinger liquids [5]. In this case, the chiral density n_+ can be taken to describe the right and left moving pieces of one semi-infinite Luttinger liquid. The appropriate term which tunnels an electron (charge e) through the junction is then

$$\mathcal{H}_{\text{tunn}} = -t_e \cos[(\phi_+ - \phi_-)/g]\delta(x). \qquad (10)$$

To proceed we first define new fields [13,14] which propagate in the same direction: $\phi_1(x) = \phi_+(x)$ and $\phi_2(x) = \phi_-(-x)$, and associated densities, $n_j = \partial_x\phi_j$ with $j = 1, 2$. One can then define commuting even and odd densities, $n = n_1 - n_2 = \partial_x\phi/2\pi$ and $N = n_1 + n_2$. The full Hamiltonian with backscattering factorizes,

$$\mathcal{H} = (\pi v/2g)(n^2 + N^2) - t_B \cos\phi\,\delta(x). \qquad (11)$$

The backscattered electrical current is given by

$$J = \int_x \partial_t n/2 = g t_B \sin\phi(x = 0), \qquad (12)$$

where the second equality follows from commuting n with the Hamiltonian. For the case of a tunnel junction, the tunnel current is $J = t_e \sin[\phi(x = 0)/g]$. Similarly, the backscattered thermal current can be written

$$J_Q = \int_x \partial_t(\mathcal{H}_0^+ - \mathcal{H}_0^-)/2 = (\pi v/g)N(x = 0)J, \qquad (13)$$

where again the time derivatives are evaluated by commuting with the Hamiltonian. This form also holds for the tunnel junction. Notice that the thermal current has been decomposed into a product of two commuting contributions: the even density N and the electrical current J which depends only on the odd boson. This remarkable simplification enables us to derive an expression relating the thermal and electrical conductances.

To this end consider the current correlation function,

$$P^R(t) = i\Theta(t)\langle[J(t), J(0)]\rangle = i\Theta(t)[P^>(t) - P^<(t)], \quad (14)$$

from which the electrical conductance follows:

$$\text{Re}G(\omega) = \frac{1}{\omega}\text{Im}P^R(\omega) = \frac{1}{2\omega}[P^>(\omega) - P^<(\omega)]. \quad (15)$$

Denoting the corresponding correlators for the thermal current, J_Q, with a subscript Q, the thermal conductance can be obtained from

$$K = \lim_{\omega \to 0}\frac{1}{\omega T}\text{Im}P_Q^R(\omega) = \frac{1}{2T^2}P_Q^<(\omega = 0), \quad (16)$$

where the latter equality follows upon using the detailed balance relation, $P_Q^>(\omega) = \exp(\beta\omega)P_Q^<(\omega)$.

The relation between the thermal and electrical current operators (13) allows us to relate their respective correlation functions,

$$P_Q^<(t) = (\pi v/g)^2 D_2^<(t)P^<(t), \quad (17)$$

where D_2 is the even boson density-density $(N - N)$ correlation function. Using the fact $D_2^<(t) = D_2^>(-t)$ allows one to express the thermal conductance as

$$K = \frac{1}{2}\left(\frac{v\pi}{gT}\right)^2\int\frac{d\omega}{2\pi}D_2^>(\omega)P^<(\omega). \quad (18)$$

The function $D_2^>$ can be readily extracted since $\mathcal{H}$ in (11) is quadratic in N, giving $D_2^>(\omega) = (g/\pi v^2)\omega e^{\beta\omega}b_\omega$. In addition, using (15), along with detailed balance, relates $P^<$ to the electrical conductance: $P^<(\omega) = 2\omega b_\omega\text{Re}G(\omega)$. We thereby obtain our final expression relating the thermal and electrical conductances through the impurity,

$$K = \frac{1}{8gT^2}\int d\omega\frac{\omega^2\text{Re}G(\omega)}{\sinh^2(\omega/2T)}. \quad (19)$$

In the absence of any backscattering, $\text{Re}G(\omega) = g/2\pi$, which gives the pure result $K = (\pi^2/3)k_B^2T/h$. In the limit of strong backscattering, K can be obtained from (18) by calculating $P^<(\omega)$ perturbatively in powers of t_e, the electron amplitude to tunnel through the junction. To leading order, P is the correlation function of the current, $J = t_e\sin(\phi/g)$, evaluated with the free odd boson Hamiltonian [i.e., (11) with $t_B = 0$]. Explicitly, $P^<(t) = ct_e^2(\pi T/\sinh\pi tT)^{2/g}$, where c is a nonuniversal constant depending on a short-time cutoff. The Fourier transform of (17) may then be computed exactly for $\omega = 0$, leading to $K = c't_e^2T^{2/g-1}$, where $c' = \pi c/4gB(1/2, 1/g + 1)$ and B is the beta function. Since c also enters in G, it drops out in the Lorentz ratio, which in this limit is found to be

$$L = 3L_0/(2g + g^2). \quad (20)$$

For noninteracting electrons $(g = 1)$ this reduces to the Fermi-liquid value, but with repulsive interactions $(g < 1)$, is larger.

As one lowers the temperature, the Lorentz number crosses over from the pure value, $L = L_0/g$, to the strong backscattering value (20). For the special case of $g = 1/2$, an expression for this crossover can be obtained explicitly. In particular, when $g = 1/2$, a closed form expression for the ac conductance follows from the exact solution [5]:

$$\text{Re}G(\omega) = \frac{g}{2\pi\omega}\int dE(f_E - f_{E+\omega})\frac{E^2}{E^2 + T_B^2}, \quad (21)$$

with $f_\omega = (e^{\beta\omega} + 1)^{-1}$. Here T_B is a crossover temperature scale, $T_B \sim t_B^{1/(1-g)} \sim t_B^2$ (for $g = 1/2$). Together (19) and (21) allow one to compute the Lorentz number for arbitrary T/T_B. For $T \gg T_B$ one finds $L = 2L_0$ in agreement with the pure result (7), but for $T \ll T_B$, the result is $L = 18L_0/5$—a factor of $3/2$ *larger* than the strong backscattering result (20). This discrepancy can be traced to an irrelevant operator ignored in the perturbative calculation leading to (20), but included implicitly in the $g = 1/2$ crossover. Specifically, consider a perturbation coupling the electron densities across the junction in the strong backscattering limit

$$\mathcal{H}_{\text{pert}} = a\delta(x)n_+n_-. \quad (22)$$

This term does not transfer charge across the junction, but does transfer energy and so contributes to thermal conduction. Moreover, it feeds into the ac electrical conductance [15] as $\text{Re}G(\omega) \sim a^2\omega^2$. Insertion into (19) then gives a contribution to the thermal conductance varying as $K \sim a^2T^3$, which must be added to the $T^{2/g-1}$ term coming from electron tunneling.

Three cases should then be distinguished, as sketched in Fig. 1. For $g > 1/2$ the electron tunneling term dominates the thermal transport at low temperatures, and (20) is correct as $T \to 0$. For the soluble case $g = 1/2$ both processes vary as T^3 and contribute to the Lorentz number as $T \to 0$. Since the coefficient "a" depends on details of the junction, a nonuniversal Lorentz number is predicted in this case. However, since the contribution to K from (22) is positive (proportional to a^2), the Lorentz
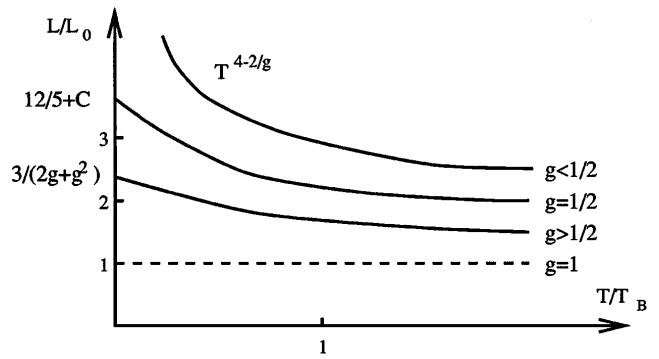
FIG. 1.   Lorentz number L for transport though a single impurity versus temperature, for several values of g.

number should be bounded below by (20): $K = 12/5 + C$ with $C > 0$, consistent with the exact solution. Finally, for $g < 1/2$, the interaction term (22) dominates K at low temperatures and the Lorentz number *diverges* as $T \to 0$,

$$L \sim (a^2 T^3)/T^{2/g-1} \sim a^2 T^{4-2/g}. \tag{23}$$

Physically, the inclusion of strong electron interactions enables heat to be transported across the junction much more readily than charge.

To compute the thermopower associated with transport through the impurity it is necessary to include in the Hamiltonian (11) terms which break particle-hole symmetry. In addition to bulk cubic interactions arising from dispersion, another local term is of the form $N \cos(\phi)\delta(x)$, which arises from an energy dependence of the matrix element t_B. One finds a thermopower linear in temperature, $Q = cT$, with a nonuniversal coefficient.

The present results can readily be generalized to include the electron spin degree of freedom. Assuming SU(2) spin symmetry, the Luttinger liquid can be characterized by a dimensionless charge conductance g_ρ, which is equal to 2 for noninteracting electrons [5]. For $g_\rho > 2/3$ the Lorentz number L crosses over from $L_0(2/g_\rho)$ to $3L_0(g_\rho + 2)^2/8g_\rho(g_\rho + 1)$ as one lowers the temperature and scales from the weak to strong backscattering regimes. For $g_\rho < 2/3$, L diverges as T^{3-2/g_ρ} in the low temperature limit. Thermopower in a Luttinger liquid with spin has also recently been discussed in Ref. [16].

Many impurities.—When many impurities are present in a repulsively interacting 1D Luttinger liquid, the conducting state is unstable to localization. However, in systems with sufficiently dilute but strong scatterers, there will be a range of temperatures over which the above single impurity results should be observable [17]. Specifically, consider a model of many 1D conductors, which are coupled electrically by dilute weak links, with a typical large separation L. These weak links would serve as "impurities" in the 1D transport. Such a model might be appropriate for some conducting polymer systems, in which the polymer backbones provide the 1D conduction channel, and the weak links arise from tunneling between different polymers at their ends. Applicability of this model requires the temperature to be above the tunneling rate, $t_\perp$, between parallel neighboring polymers. Otherwise the transport would no longer be 1D, but would take place coherently across many chains. Moreover, we require that tunneling events across successive weak links be incoherent, so that $k_B T > \hbar v/L$, where v is the 1D Fermi velocity. In this temperature regime, the bulk electrical conductivity should vary as a power law, $\sigma(T) \sim T^{2/g-2}$, before crossing over at lower temperatures into either a localized regime or a bulk metallic state. Our single impurity results imply that for $g > 1/2$ the thermal conductivity should also vary as a power law, $\kappa \sim T^{2/g-1}$, with a universal non-Fermi-liquid Lorentz

number given by (20). For $g < 1/2$ an even larger (and diverging) Lorentz ratio is predicted.

In some conducting polymer samples, the electrical conductivity does vary as a power law with temperature [18]. This power law has been interpreted [18] as being in the vicinity of a bulk metal-insulator transition. However, at a 3D Anderson localization transition a temperature independent thermopower is predicted [19], in contrast to the measured behavior, $Q \sim T$. In a model of 1D conductors with dilute weak links, a linear metallic thermopower would be expected. The Lorentz number provides a further difference between these two models. At the Anderson transition the Lorentz number is predicted to be suppressed [19] below the Fermi-liquid value L_0 by roughly $2/3$, whereas our results show an enhanced Lorentz number for 1D thermal transport through dilute impurities. It would be most interesting to measure thermal conductivity in conducting polymer samples which exhibit power law electrical conductivities.

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