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Quantum Brownian motion in a periodic potential and the multichannel Kondo problem

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We study the motion of a particle in a periodic potential with Ohmic dissipation. In $D=1$ dimension it is well known that there are two phases depending on the dissipation: a localized phase with zero temperature mobility $\mu=0$ and a fully coherent phase with μ unaffected by the periodic potential. However, for nonsymmorphic lattices with $D>1$, such as the honeycomb lattice, there is an intermediate phase with a universal mobility μ^*. This intermediate phase is relevant to resonant tunneling experiments in strongly coupled Coulomb-blockade structures as well as multichannel Luttinger liquids. We relate this problem to the Toulouse limit of the $D+1$ channel Kondo problem, which allows us to compute μ^* exactly using results known from conformal field theory. [S0163-1829(98)52110-8]

The quantum mechanics of a particle in a periodic potential coupled to a dissipative environment is a fundamental problem.[1] A simple theory based on the Caldeira-Leggett model of Ohmic dissipation was proposed in the mid 1980s as a possible description of the motion of a heavy charged particle in a metal.[2] Recently there has been renewed interest in this quantum Brownian motion (QBM) model in connection with quantum impurity problems[3,4] and boundary conformal field theory.[5] Kane and Fisher[3,4] have shown that this model describes tunneling through a barrier in a single-channel Luttinger liquid (LL), which is relevant to experiments on quantum wires[6] and quantum Hall edge states.[7] In this mapping the QBM takes place in an abstract space where the ''coordinate'' of the ''particle'' is the number of electrons that tunnel past the barrier. The periodic potential is due to the discreteness of the electron's charge. Low-energy electronic excitations play the role of the dissipative bath. There are two possible zero-temperature ($T=0$) phases: a localized phase in which the conductance $G=0$, and a fully coherent phase with ''perfect'' conductance.

A related problem concerns the Coulomb blockade in quantum dots connected to leads via quantum point contacts (QPC). Provided the level spacing ΔE of the quantum dot is sufficiently small, Furusaki and Matveev (FM)[8] argued that each QPC may be treated as an independent one-dimensional (1D) system. For spin-1/2 electrons, the dot may be described by four coordinates: the number of electrons of each spin transfered through each of the two leads. Since the Coulomb blockade constrains the total charge, this maps to QBM on a *three-dimensional* potential.

FM studied resonances in such a system with two identical QPC's.[8] They showed that the conductance on resonance is *less* than the perfect conductance e^2/h.[9] While they did not calculate its precise value, they argued that it is universal and is controlled by the fixed point (FP) of the four-channel Kondo problem (KP). This prediction is particularly interesting in light of recent experiments on gated quantum dots.[10] Similar FP's with intermediate conductance were found in models of resonant tunneling in a spin-1/2 LL[4] and in conformal field theory.[11] Analysis of these FP's, however, was only possible in certain special limits.

In this paper we consider the general problem of QBM on periodic lattices. The lattice symmetry plays a crucial role in determining the $T=0$ phases. For the honeycomb lattice and its $N-1$ dimensional generalization, there is a $T=0$ phase described by a FP which we relate to the N channel KP. Exploiting the mapping onto the KP, we compute exactly the FP mobility using results from conformal field theory. This allows us to compute exactly the resonant conductance in both in the FM model, and in a LL with a particular interaction strength. Moreover, this analysis provides a unified framework for understanding the nature of the intermediate FP's.

The QBM model is most easily described using an imaginary time path integral for the coordinate $\mathbf{r}(\tau)$. In 1D electronic problems, this is derived by first bosonizing, and then integrating out the degrees of freedom away from the QPC.[3] The action has the form[1]

$$S=S_0[\mathbf{r}(\tau)]-\int \frac{d\tau}{\tau_c}\sum_{\mathbf{G}} v_{\mathbf{G}} e^{i2\pi\mathbf{G}\cdot\mathbf{r}(\tau)}, \tag{1}$$

where $v_{\mathbf{G}}$ are Fourier components of the periodic potential at the reciprocal-lattice vectors $\mathbf{G}$ (defined so that $\mathbf{G}\cdot\mathbf{R}$ is an integer for any lattice vector $\mathbf{R}$). The coupling to the dissipative bath gives

$$S_0[\mathbf{r}(\tau)]=\frac{1}{2}\int d\omega|\omega|e^{|\omega|\tau_c}|\mathbf{r}(\omega)|^2, \tag{2}$$

where τ_c is a short time cutoff. Although the coefficient of S_0 determines the amount of dissipation, it may be fixed by rescaling $\mathbf{r}$ and $\mathbf{G}$. The dissipation is thus controlled by the lattice constant R_0. Note that Eqs. (1) and (2) have *exactly* the same form as the bosonized actions in Refs. 3 and 4 for LL's, and also in Ref. 8 for QPC's after integrating out degrees of freedom away from $x=0$. A QPC with noninteracting electrons is described by a 1D lattice with $R_0=1$. For a LL with repulsive interactions, $R_0>1$.[3] We define the dimensionless mobility

$$\mu=\frac{2\pi}{D}\lim_{\omega\to 0}|\omega|\langle|\mathbf{r}(\omega)|^2\rangle. \tag{3}$$

μ describes the average velocity of the particle in response to a uniform applied force, normalized such that $\mu=1$ when $v_{\mathbf{G}}=0$.

The effect of the periodic potential may be analyzed perturbatively in either of two limits. A weak potential may be studied by considering the renormalization group (RG) flows to leading order in $v_{\mathbf{G}}$,

$$dv_{\mathbf{G}}/d\ell=(1-|\mathbf{G}|^2)v_{\mathbf{G}}. \tag{4}$$

Clearly, if the shortest reciprocal-lattice vector satisfies $|\mathbf{G}_0|>1$, then all $v_{\mathbf{G}}$ are irrelevant. The ''small barrier'' limit in which $\mu=1$ is thus perturbatively stable. On the other hand, if $|\mathbf{G}_0|<1$, then the system flows to a different strong coupling phase.

When the barriers are large, the particle is localized in one of the minima of the potential with a small probability for tunneling to another. It is then more natural to consider a dual representation in which the partition function is expanded in powers of the fugacity of these tunneling events.[2,3] For a Bravais lattice, this may be generated by expanding the dual action,

$$S=S_0[\mathbf{k}(\tau)]-\int\frac{d\tau}{\tau_c}\sum_{\mathbf{R}}t_{\mathbf{R}}e^{i2\pi\mathbf{R}\cdot\mathbf{k}(\tau)}. \tag{5}$$

$t_{\mathbf{R}}$ may be interpreted as the matrix element for the particle to tunnel between minima connected by a lattice vector $\mathbf{R}$. Equivalently, $\mathbf{k}(\tau)$ describes the particle's trajectory in momentum space in a potential with the symmetry of the reciprocal lattice. The RG flows to leading order in $t_{\mathbf{R}}$ are then

$$dt_{\mathbf{R}}/d\ell=(1-|\mathbf{R}|^2)t_{\mathbf{R}}. \tag{6}$$

The ''large barrier'' limit is thus perturbatively stable provided the shortest lattice vector satisfies $|\mathbf{R}_0|>1$.

For a one-dimensional lattice, $|\mathbf{R}_0||\mathbf{G}_0|=1$. Thus either the small or the large barrier limit is stable, but not both. There are two phases: for $|\mathbf{G}_0|<1$ the system is localized and for $|\mathbf{G}_0|>1$ the system has perfect mobility. Clearly, this is also the case in higher dimensions for a lattice with cubic symmetry.

In contrast, for a triangular lattice, $|\mathbf{R}_0||\mathbf{G}_0|=2/\sqrt{3}$. It follows that for $1<|\mathbf{G}_0|^2<4/3$, *both* the small and large barrier limits are stable. There must therefore be an unstable FP separating the two phases, as indicated in Fig. 1(a). A similar intermediate FP occurs in the single barrier problem of a spin-1/2 LL.[4] An analysis perturbative in $v_{\mathbf{G}}$ of this FP is possible for $|\mathbf{G}_0|^2=1+\epsilon$. Let $v_{\mathbf{G}}=v$ for the six nearest-neighbor reciprocal lattice vectors. For $v>0$, the potential minima form a triangular lattice. The RG flow to second order in v is[4]

$$dv/d\ell=-\epsilon v+2v^2. \tag{7}$$

Provided $v>0$ and $\epsilon>0$, there is an unstable FP $v^*=\epsilon/2$. The mobility at this FP is universal, $\mu^*=1-(3\pi^2/2)\epsilon^2$. A similar analysis for small t at $|\mathbf{R}_0|^2=1+\epsilon$ gives $\mu^*=(3\pi^2/2)\epsilon^2$. For $|\mathbf{G}_0|^2=|\mathbf{R}_0|^2=2/\sqrt{3}$, the theory is self dual, which implies that $\mu^*=1/2$. Piecing these results together, we obtain the flow diagram in Fig. 1(a).

When $v<0$, the minima of the potential described above form a honeycomb lattice. The honeycomb lattice is equivalent to the triangular lattice described above with a two-site basis. While the reciprocal lattice is still triangular, $|\mathbf{R}_0|$ is shorter, and $|\mathbf{G}_0||\mathbf{R}_0|=2/3$. Thus, for $4/9<|\mathbf{G}_0|^2<1$ both the large and small barrier limits are *unstable*, so that there must be a stable FP describing a new intermediate coupling phase.

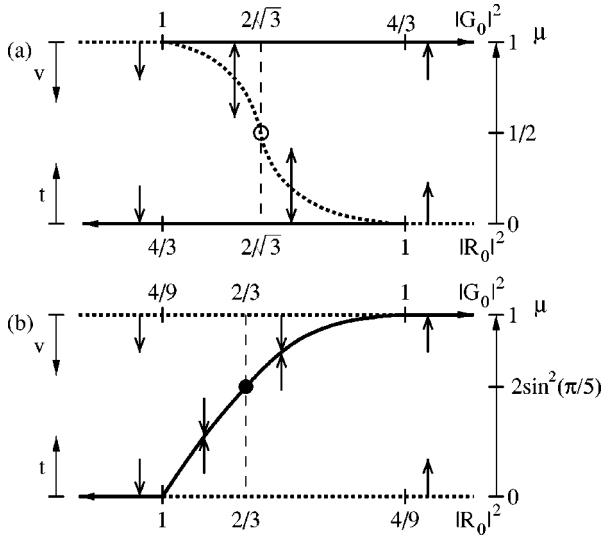

FIG. 1. Flow diagrams for the (a) triangular lattice and (b) honeycomb lattice as a function of lattice constant. The top (bottom) lines represent the small v (t) limits. Stable (unstable) fixed points are depicted by solid (dotted) lines, and arrows indicate the RG flows. The mobility μ is indicated by the vertical axis on the right.

A perturbative analysis is again possible in the large and small barrier limits. For small barriers the FP of Eq. (7) is stable for $v<0$ and $\epsilon<0$. μ^* is the same as above. In the large barrier theory we must keep track of the two-site basis of the honeycomb lattice. There are three nearest neighbors $\mathbf{R}$ for each site on the A sublattice. For the B sublattice the nearest neighbors are $-\mathbf{R}$. The tunneling must alternate between the sublattices. This can be incorporated in the dual theory by introducing a spin-1/2 degree of freedom. For nearest-neighbor hopping, the dual action is given by

$$S=S_0[\mathbf{k}]-\int\frac{d\tau}{\tau_c}\sum_{\mathbf{R}}t[\tau^+e^{i2\pi\mathbf{R}\cdot\mathbf{k}}+\tau^-e^{-i2\pi\mathbf{R}\cdot\mathbf{k}}], \tag{8}$$

where $\mathbf{R}$ are among the three nearest-neighbor lattice vectors of sublattice A and $\tau^{\pm}$ are spin-1/2 operators, $\sigma^{\pm}/2$. A perturbative analysis for $|\mathbf{G}_0|^2=1+\epsilon$ gives $\mu^*=\pi^2\epsilon$.

The flow diagram for the honeycomb lattice as a function of lattice constant is summarized in Fig. 1(b). Unlike the cubic and triangular Bravais lattices, the $T=0$ mobility does not exhibit a discontinuous jump from 0 to 1. Rather, the mobility interpolates smoothly between the two limits in the intermediate phase for $4/9<|\mathbf{R}_0|^2<1$. In general, the existence of a stable intermediate phase requires a nonsymmorphic lattice symmetry, with a vector connecting equivalent sites that is shorter than any lattice translation.

We now relate the stable intermediate FP to the multichannel KP by identifying the lattice symmetry in the KP. The Hamiltonian of the anisotropic N channel KP is[12]

$$\mathcal{H}=iv_F\sum_{a,s}\int dx\psi^{\dagger}_{as}\partial_x\psi_{as}+2\pi v_F\sum_{i,a}J_iS^i_{\mathrm{imp}}s^i_a(0),$$

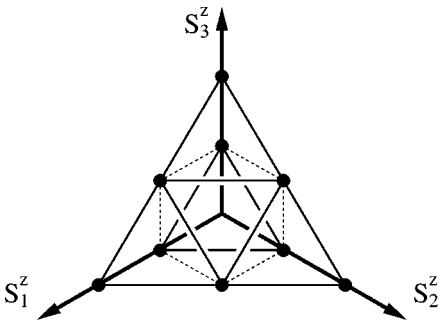

FIG. 2. Lattice of spin states for the three-channel KP, which form two planes with constant $S_1^z+S_2^z+S_3^z$.

where a, s, and i are channel, spin, and space indices, $S_{\rm imp}^i$ is the impurity spin, and $s_a^i(0)=\psi_{as}^\dagger(0)(\sigma_{ss'}^i/2)\psi_{as'}(0)$ is the electronic spin in channel a at $x=0$. We consider an anisotropic model, characterized by dimensionless couplings J_z and $J_x=J_y=J_\perp$. Our analysis closely parallels that of Emery and Kivelson for the two-channel KP.[13] We first bosonize the theory, and then do a rotation in spin space which transforms the J_z interaction. Upon integrating out the degrees of freedom away from $x=0$, we obtain a theory in terms of the boson fields at the impurity which closely resembles the lattice models studied in this paper. The details of this mapping will be presented in a longer article, however its essence may be understood quite simply.

Now the total spin S_a^z in each of the N channels plays the role of the coordinates of the QBM model. When $J_\perp=0$, the state of the system may be characterized by a vector of discrete values of S_a^z, which form an N-dimensional cubic lattice. $J_\perp$ ''hops'' the system between sites on this lattice. Since $\mathcal{H}$ conserves the total spin of the electrons plus the impurity, the system is constrained to lie on one of two lattice planes with constant $S_{\rm imp}^z+\Sigma_a S_a^z$ where $S_{\rm imp}^z=\pm 1/2$. For $N=3$ each lattice plane forms a triangular lattice, as sketched in Fig. 2. Viewed from the (111) direction, the two lattice planes form a ''corrugated'' honeycomb lattice in which the two triangular sublattices are displaced in the perpendicular direction. For general N, the lattice planes consist of two interpenetrating $N-1$ dimensional close packed lattices. For $N=4$, they form a corrugated diamond lattice.

Now consider QBM on such a lattice described by

$$S=S_0[\mathbf{k}]-\int\frac{d\tau}{\tau_c}\sum_{\mathbf{R}_\parallel} t[\tau^+ e^{i2\pi(\mathbf{R}_\parallel\cdot\mathbf{k}_\parallel+R_\perp k_\perp)}+\text{H.c.}].$$

For $N=3$, $\mathbf{k}$ is a three-dimensional vector with components $\mathbf{k}_\parallel$ and $k_\perp$ parallel and perpendicular to the lattice plane. $\mathbf{R}_\parallel$ are chosen from the three nearest-neighbor lattice vectors for the honeycomb lattice, and $R_\perp$ is the perpendicular displacement between the two sublattices shown in Fig. 2. $\tau^\pm$ guarantee that the hopping alternates between the two lattice planes. Now we replace $\mathbf{r}$ with S_a^z. Then, this is identical to the N-channel KP with $t=J_\perp/2$, provided the lattice constants are chosen to give the appropriate scaling for $J_\perp$. For $J_z=0$, the dimension of $s_a^+(0)$ is 1, so the cubic lattice constant in Fig. 2 is 1. It follows that $|\mathbf{R}_\parallel|^2=1-1/N$. Finite J_z may be treated nonperturbatively using bosonization,[13] and affects the dimension of $s_a^+(0)$. This leads to a distortion

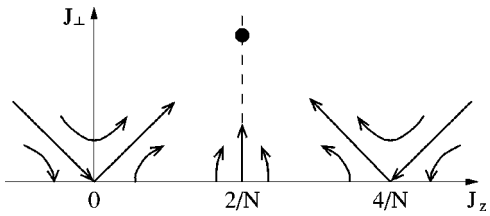

FIG. 3. Flow diagram for the N-channel Kondo model for small $J_\perp$. The dashed line is the Toulouse limit, $J_z=2/N$. The strong-coupling fixed point is marked with the full circle.

of the lattice in the perpendicular direction, $R_\perp^2=(1-NJ_z/2)^2/N$. Note that $R_\perp=0$ for $J_z=2/N$, so that the perpendicular direction decouples. This is the N-channel generalization of the Toulouse limit.[13–15] A central point of this paper is that this limit of the N-channel KP is *identical* to a QBM model on a $N-1$ dimensional honeycomb lattice.

An RG analysis shows that the corrugation of the lattice, given by $R_\perp$, is irrelevant. The flow equations, expanded to order $J_\perp^3$ are[16,17]

$$dJ_z/d\ell=J_\perp^2[1-(N/2)J_z], \tag{9}$$

$$dJ_\perp/d\ell=J_\perp J_z[1-(N/4)J_z]-(N/4)J_\perp^3. \tag{10}$$

As shown in Fig. 3, J_z flows towards $2/N$, the Toulouse limit, shown by the dashed line. For $N=3$ this is the same as the dashed line in Fig. 1(b). The intermediate FP for the honeycomb lattice with $|\mathbf{R}_0|^2=2/3$ is the same as that of the three-channel KP. Varying $|\mathbf{R_0}|$ adiabatically connects the multichannel Kondo FP to the strong and weak barrier limits described above. For large N, the FP at $J_\perp=J_z=2/N$ is in the strong barrier limit.[18]

Conformal field theory allows for an exact description of the multichannel Kondo FP.[5] This allows us to compute μ^* nonperturbatively, by identifying the appropriate correlation function in the KP. The analog of r_a is the spin in each channel, S_a^z. μ then describes the flow of spin between the channels in response to an applied potential. Introducing an orthogonal transformation $r_A=O_{Aa}r_a$, $\mathbf{r}$ may be projected onto the $N-1$ dimensional plane with $\Sigma_a r_a$ constant (see Fig. 2). For $A=1,\ldots,N-1$ this has the form $O_{Aa}=T_{aa}^A$, where T^A are the diagonal generators of SU(N) normalized such that $\text{Tr}[T^AT^B]=\delta_{AB}$. In terms of electronic operators we then have, $r_A=\int dx\psi_{as}^\dagger(\sigma_{ss'}^z/2)T_{aa'}^A\psi_{a's'}$. Ludwig and Affleck[19] have computed all correlation functions of r_A. Using Eq. (3) and borrowing their results, we obtain,

$$\mu^*=2\sin^2\frac{\pi}{N+2}. \tag{11}$$

For $N=2$ the FP is at the small barrier limit $\mu^*=1$. For $N=3$, μ^* is plotted in Fig. 1(b).

We now return to the FM theory of Coulomb-blockade resonances, which is described by a four dimensional QBM model, with the coordinates $\mathbf{r}$ replaced by $Q_{i\sigma}$, the charge in lead i with spin σ. Consider the limit of strong barriers, when $Q_{i\sigma}$ are constrained to be integers. Off resonance, the charge on the dot has a fixed value, which implies $\Sigma_{i\sigma}Q_{i\sigma}$ is fixed. The allowed values of $Q_{i\sigma}$ then form a fcc lattice with $|\mathbf{R}_0|^2=2$. On resonance, the charge on the dot can fluctuate

between two values. In a manner similar to that shown in Fig. 2, this leads to a *diamond* lattice with $|\mathbf{R}_0|^2=3/4$. Our analysis allows us to identify the universal resonant conductance. From linear response theory, $G^*=\lim_{\omega\to 0}(e^2/\hbar)|\omega|\langle|Q_{1\uparrow}+Q_{1\downarrow}|^2\rangle$. Using the transformation O_{Aa} and Eq. (3) we obtain $G^*=(e^2/h)\mu^*$, with $\mu^*=1/2$ from Eq. (11).

Away from resonance, the conductance has the scaling form $G=\tilde{G}(\delta/T^{2/3})$ where δ is a tuning parameter, such as a gate voltage. $\tilde{G}$ is a universal scaling function which describes the crossover from the four-channel Kondo FP (diamond lattice) to the insulating FP (fcc lattice) when the sublattice symmetry is broken.

Our analysis also applies to resonant tunneling through a *single* resonant state (the limit where $\Delta E\gg T$) when the leads are LL's.[4] Resonant tunneling in a single channel LL may be described by QBM on a corrugated 1D lattice, in which the two equivalent 1D sublattices correspond to the occupation of the resonant state. When the interaction parameter $g=1/2$, the lattice constant is such that this maps to the two-channel KP. Other values of g correspond to different lattice constants and are related adiabatically to the two-channel KP. The universal resonance line shape has been computed for both $g=1/2$ and $g=1/3$.[4,20] Perhaps similar techniques could be used to calculate the line shape for the FM model.

A similar analysis may be applied to a spin-degenerate LL. Tunneling across a barrier then corresponds to QBM on a *two*-dimensional lattice. The lattice constants are determined by the interaction parameters $g_{\rho,\sigma}$ in the charge and spin sectors. SU(2) spin symmetry constrains $g_\sigma=2$, whereas repulsive interactions reduce g_ρ below the noninteracting value of 2. In Ref. 4 it was shown that resonances can be reached by tuning a single parameter, such as a gate voltage which controls the occupation of the resonant state. For strong interactions ($g_\rho<1$), the resonances have intermediate conductance. For the particular values $g_\rho=2/3$, $g_\sigma=2$, the lattice has triangular symmetry off resonance. On resonance the symmetry is that of a honeycomb lattice with $|\mathbf{R}_0|^2=2/3$. The resonance FP thus corresponds precisely to the three-channel Kondo FP, and the resonant conductance is $G^*=(2/3)(e^2/h)\mu^*$ with $\mu^*=2\sin^2(\pi/5)$. For other values of g_ρ the lattice is distorted, and resonances are related adiabatically to the three-channel KP.

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