Nonlinear Elasticity

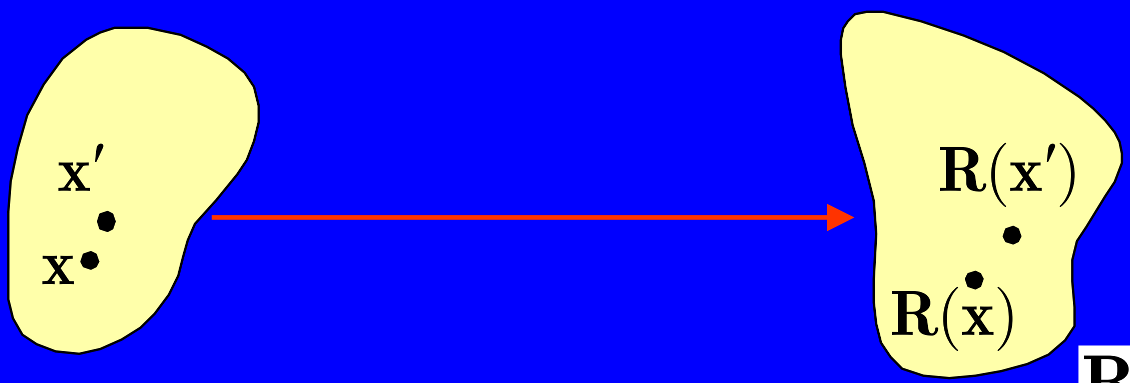
Outline

- Some basics of nonlinear elasticity
- Nonlinear elasticity of biopolymer networks
- Nematic elastomers

What is Elasticity

- Description of distortions of rigid bodies and the energy, forces, and fluctuations arising from these distortions.
- Describes mechanics of extended bodies from the macroscopic to the microscopic, from bridges to the cytoskeleton.

Classical Lagrangian Description



$$\mathbf{R}(\mathbf{x}) = \mathbf{x} + \mathbf{u}(\mathbf{x})$$

Reference material in D dimensions described by a continuum of mass points x. Neighbors of points do not change under distortion

Material distorted to new positions R(x)

$$\Lambda_{i\alpha} = \frac{\partial R_i}{\partial x_\alpha} = \delta_{i\alpha} + \eta_{i\alpha}$$

Cauchy deformation tensor

$$\eta_{i\alpha} = \partial_\alpha u_i$$

Linear and Nonlinear Elasticity

Linear: Small deformations – Λ near 1

Nonlinear: Large deformations – Λ >>1

Why nonlinear?

- Systems can undergo large deformations – rubbers, polymer networks , …
- Non-linear theory needed to understand properties of statically strained materials
- Non-linearities can renormalize nature of elasticity
- Elegant an complex theory of interest in its own right

Why now:

- New interest in biological materials under large strain
- Liquid crystal elastomers – exotic nonlinear behavior
- Old subject but difficult to penetrate – worth a fresh look

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Deformations and Strain

Complete information about shape of body in $\mathbf{R}(\mathrm{x}) = \mathbf{x} + \mathbf{u}(\mathbf{x})$;

$\mathbf{u} = \mathrm{const.}$ – translation no energy.

No energy cost unless $\mathbf{u}(\mathbf{x})$ varies in space.

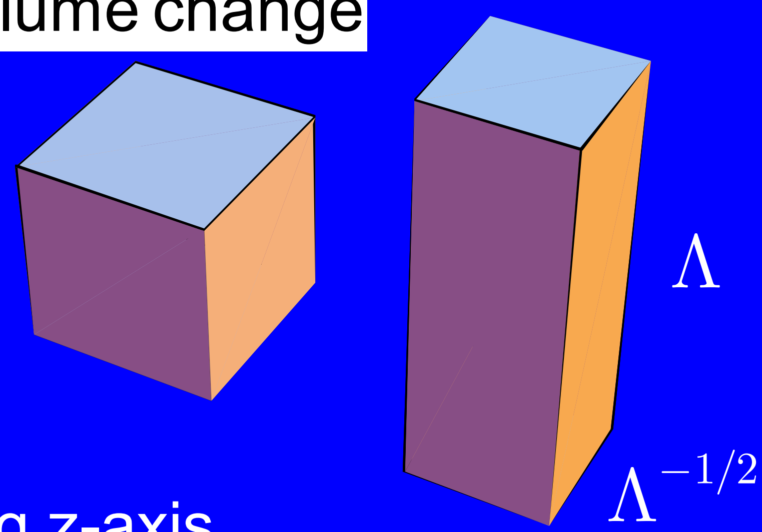
For slow variations, use the Cauchy deformation tensor

$$\boxed{\Lambda_{i\alpha} = \delta_{i\alpha} + \partial_\alpha u_i = \delta_{i\alpha} + \eta_{i\alpha}}$$

$$d^3R = \det \underset{\sim}{\Lambda}\, d^3x$$

$\det \underset{\sim}{\Lambda} = 1:$ No volume change

$$\underset{\sim}{\Lambda} = \begin{pmatrix} \Lambda^{-1/2} & 0 & 0 \\ 0 & \Lambda^{-1/2} & 0 \\ 0 & 0 & \Lambda \end{pmatrix}$$

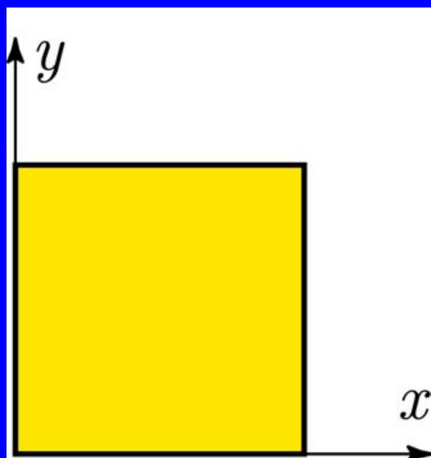


Volume preserving stretch along z-axis

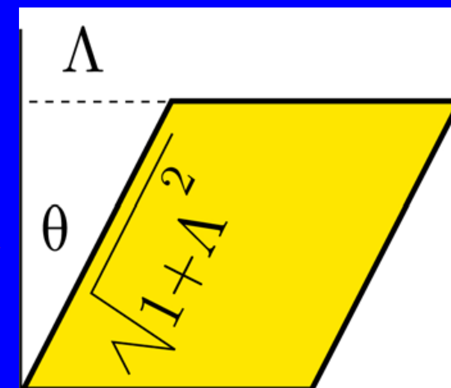
Simple shear strain

Note: Λ is not symmetric

Constant Volume, but note stretching of sides originally along x or y.

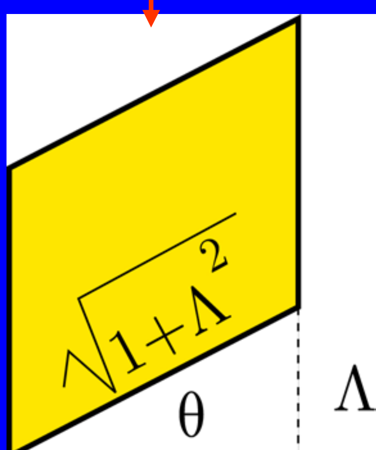


$$\underset{\sim}{\Lambda} = \begin{pmatrix} 1 & \Lambda \\ 0 & 1 \end{pmatrix}$$

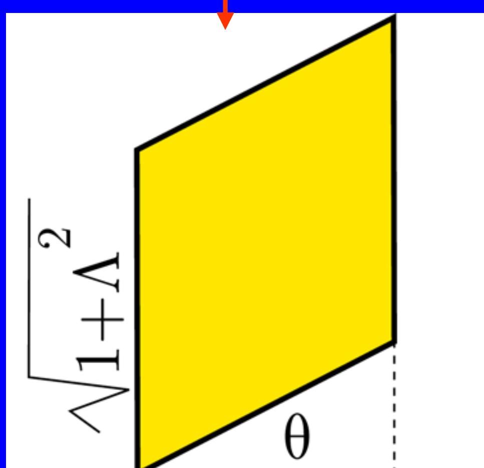


$$\underset{\sim}{\Lambda} = \begin{pmatrix} 1 & 0 \\ \Lambda & 1 \end{pmatrix}$$

Rotate



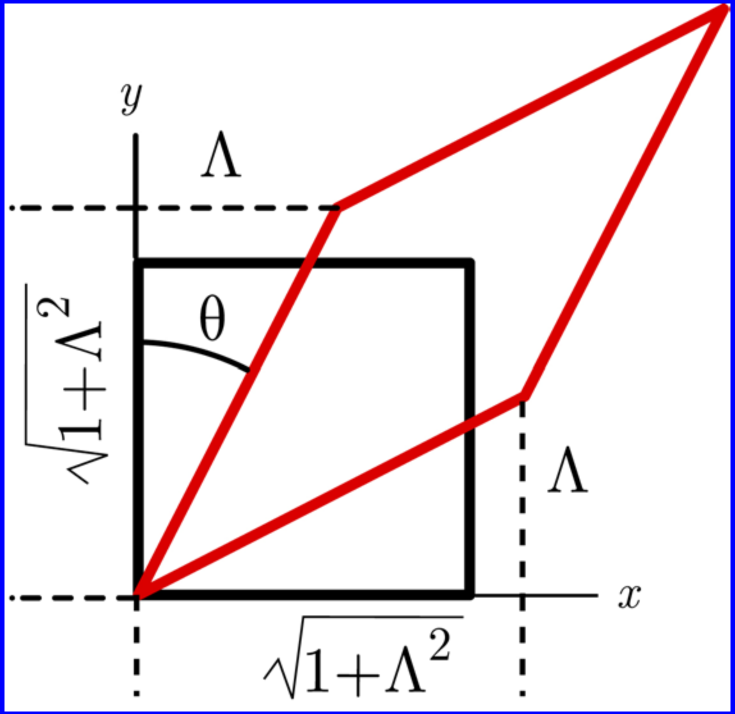
Not equivalent to



Pure Shear

Pure shear: symmetric deformation tensor with unit determinant – equivalent to stretch along 45 deg.

$$\underset{\sim}{\Lambda} = \begin{pmatrix} \sqrt{1+\Lambda^2} & \Lambda \\ \Lambda & \sqrt{1+\Lambda^2} \end{pmatrix}$$

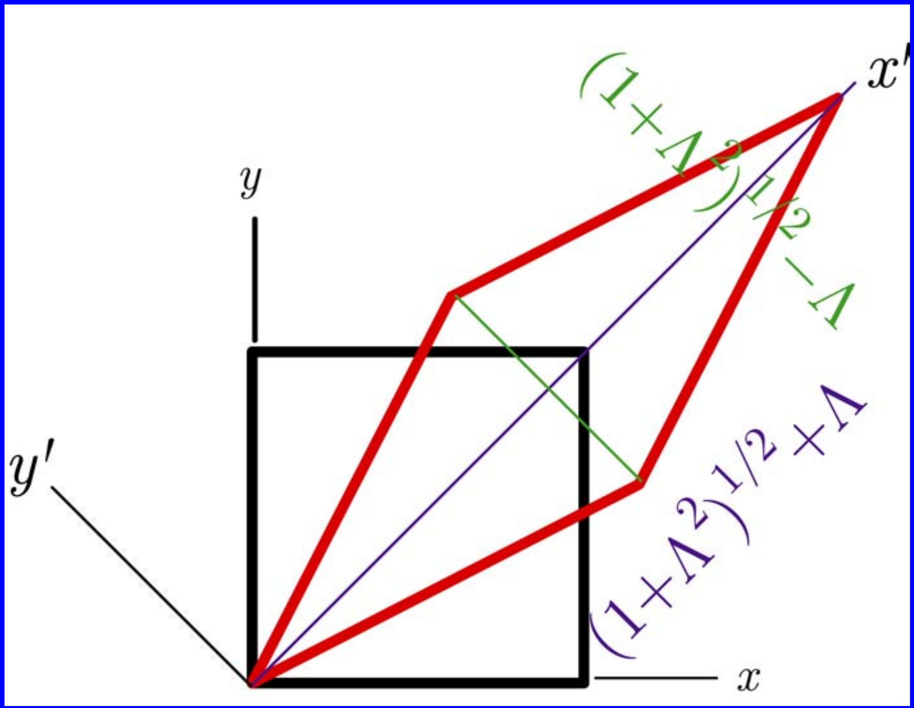


Pure shear as stretch

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \equiv U \begin{pmatrix} x \\ y \end{pmatrix}$$

$$\Lambda_{i\alpha} = \frac{\partial R_i}{\partial x_\alpha} = \frac{\partial R_i}{\partial R'_j} \frac{\partial R'_j}{\partial x'_\beta} \frac{\partial x'_\beta}{\partial x_\alpha} = U^T_{ij} \Lambda'_{j\beta} U_{\beta\alpha}$$

$$\underset{\sim}{\Lambda}' = \underset{\sim}{U}\,\underset{\sim}{\Lambda}\,\underset{\sim}{U}^T = \begin{pmatrix} \sqrt{1+\Lambda^2}+\Lambda & 0 \\ 0 & \sqrt{1+\Lambda^2}-\Lambda \end{pmatrix}$$

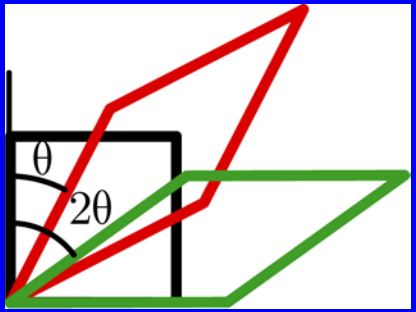


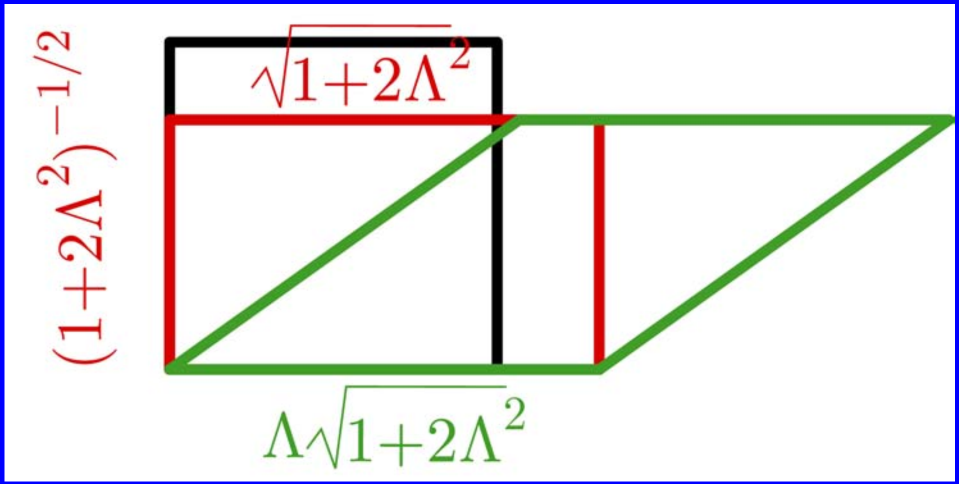
Pure to simple shear

$$\underset{\sim}{\Lambda} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} \sqrt{1+\Lambda^2} & \Lambda \\ \Lambda & \sqrt{1+\Lambda^2} \end{pmatrix}$$

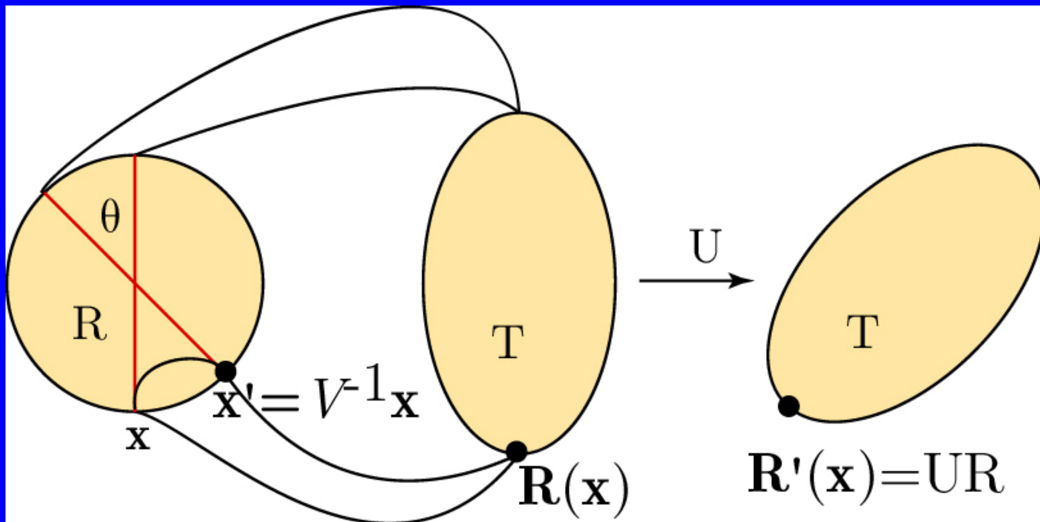
$$= \begin{pmatrix} \sqrt{1+2\Lambda^2} & 2\Lambda\sqrt{1+2\Lambda^2} \\ 0 & (1+2\Lambda^2)^{-1/2} \end{pmatrix}$$

$$\tan\theta = \frac{\Lambda}{\sqrt{1+\Lambda^2}}$$





Cauchy Saint-Venant Strain



R=Reference space

T=Target space

$$dR^2 - dx^2 = 2u_{\alpha\beta} dx_\alpha dx_\beta$$

$$\Lambda_{i\alpha} = \frac{\partial R_i}{\partial x_\alpha} = \delta_{i\alpha} + \eta_{i\alpha}$$

$u_{\alpha\beta}$ is invariant under rotations in the target space but transforms as a tensor under rotations in the reference space. It contains no information about orientation of object.

$$\underset{\sim}{u} = \tfrac{1}{2}(\underset{\sim}{\Lambda}^T \underset{\sim}{\Lambda} - \underset{\sim}{\delta}) \approx \tfrac{1}{2}(\underset{\sim}{\eta} + \underset{\sim}{\eta}^T)$$

$$u_{\alpha\beta} = \tfrac{1}{2}\left(\partial_\alpha u_\beta + \partial_\beta u_\alpha + \partial_\alpha u_k \partial_\alpha u_k\right)$$

$\boxed{\text{Symmetric!}}$

Elastic energy

The elastic energy should be invariant under rigid rotations in the target space: if is a function of $u_{\alpha\beta}$.

$$F = \tfrac{1}{2}\int d^D x f(u_{\alpha\beta})$$
$$= \tfrac{1}{2}\int d^D x [K_{\alpha\beta\gamma\delta} u_{\alpha\beta} u_{\gamma\delta} + \tilde{\sigma}_{\alpha\beta} u_{\alpha\beta}]$$

This energy is automatically invariant under rotations in target space. It must also be invariant under the point-group operations of the reference space. These place constraints on the form of the elastic constants.

Note there can be a linear “stress”-like term. This can be removed (except for transverse random components) by redefinition of the reference space

Elastic modulus tensor

$K_{\alpha\beta\chi\delta}$ is the elastic constant or elastic modulus tensor. It has inherent symmetry and symmetries of the reference space.

$$\boxed{K_{\alpha\beta\gamma\delta} = K_{\gamma\delta\alpha\beta} = K_{\beta\alpha\gamma\delta} = K_{\alpha\beta\delta\gamma}}$$

Isotropic system

$$\boxed{K_{\alpha\beta\gamma\delta} = \lambda\delta_{\alpha\beta}\delta_{\gamma\delta} + \mu(\delta_{\alpha\gamma}\delta_{\beta\delta} + \delta_{\alpha\delta}\delta_{\beta\delta})}$$

Uniaxial ($\mathbf{n}$ = unit vector along uniaxial direction)

$$\begin{aligned} K_{\alpha\beta\gamma\delta} &= C_1 n_\alpha n_\beta n_\gamma n_\delta + C_2(n_\alpha n_\beta \delta^T_{\gamma\delta} + n_{\gamma\delta} n_\beta \delta^T_{\alpha\beta}) \\ &+ C_3 \delta^T_{\alpha\beta}\delta^T_{\gamma\delta} + \tfrac{1}{2} C_4(\delta^T_{\alpha\gamma}\delta^T_{\beta\delta} + \delta^T_{\alpha\delta}\delta^T_{\beta\gamma}) + \\ &+ \tfrac{1}{4} C_5(n_\alpha n_\gamma \delta^T_{\beta\delta} + n_\alpha n_\delta \delta^T_{\beta\gamma} + n_\beta n_\delta \delta^T_{\alpha\gamma} + n_\beta n_\gamma \delta^T_{\alpha\delta}) \end{aligned}$$

Isotropic and Uniaxial Solid

Isotropic: free energy density f has two harmonic elastic constants

$$f = f(\underline{\Lambda}) = f(\underline{U}\Lambda\underline{V}^{-1})$$

$$f = f(\underline{u}) = f(\underline{V}\,\underline{u}\,\underline{V}^{-1})$$

$$= \tfrac{1}{2} B u_{\alpha\alpha}^{2} + \mu \mathrm{Tr}\,\underline{\tilde{u}}^{2} - C\,\mathrm{Tr}\,\underline{\tilde{u}}^{3} + D\left(\mathrm{Tr}\,\underline{\tilde{u}}^{2}\right)^{2}$$

Invariant under

$$\mathbf{R}(\mathbf{x}) \rightarrow \underline{\mathrm{U}}\mathbf{R}(\underline{\mathrm{V}}\mathbf{x})$$

μ = shear modulus;
B = bulk modulus

Uniaxial: five harmonic elastic constants

$$f = \tfrac{1}{2} C_1 u_{zz}^2 + C_2 u_{zz} u_{\nu\nu} + \tfrac{1}{2} C_3 u_{\nu\nu}^2 + C_4 u_{\nu\tau}^2 + C_5 u_{\nu z}^2;$$

$$\mathbf{x}_\alpha = (\mathbf{x}_\nu, x_z)$$

Invariant under

$$\mathbf{R}(\mathbf{x}) \rightarrow \underline{\mathrm{U}}\mathbf{R}(\underline{\mathrm{V}_{\mathrm{uni}}}\mathbf{x})$$

Force and stress I

$$f_i = \partial_\alpha \sigma_{i\alpha}$$

$$F^{\mathrm{ext}} = \int d^D x\, f_i u_i = -\int d^D x\, \sigma_{i\alpha} \partial_\alpha u_i$$

external force density – vector in target space. The stress tensor $\sigma_{i\alpha}$ is mixed. This is the engineering or 1st Piola-Kirchhoff stress tensor = force per area of reference space. It is not necessarily symmetric!

$$-\frac{\delta F}{\delta u_i(\mathbf{x})} = \int d^D x' \frac{\partial f}{\partial u_{\alpha\beta}(\mathbf{x}')} \frac{\delta u_{\alpha\beta}(\mathbf{x}')}{\delta u_i(\mathbf{x})} = f_i = -\partial_\alpha \sigma_{i\alpha}$$

$$\frac{\delta u_{\alpha\beta}(\mathbf{x}')}{\delta u_i(\mathbf{x})} = \tfrac{1}{2}(\Lambda_{i\alpha}\partial'_\beta + \Lambda_{i\beta}\partial'_\alpha)\delta(\mathbf{x}-\mathbf{x}')$$

$$\sigma_{i\alpha} = \Lambda_{i\beta}\frac{\partial f}{\partial u_{\beta\alpha}} \equiv \Lambda_{i\beta}\sigma^{II}_{\beta\alpha}$$

$\sigma_{\alpha\beta}^{II}$ is the second Piola-Kirchhoff stress tensor - symmetric

Note: In a linearized theory, $\sigma_{i\alpha} = \sigma_{i\alpha}^{II}$

Cauchy stress

The Cauchy stress is the familiar force per unit area in the target space. It is a symmetric tensor in the target space.

$$\int d^d x\, \sigma^I_{i\alpha} \partial_\alpha u_i = \int d^d R\, \sigma^C_{ij} \nabla_j u_i$$

$$\nabla_i \equiv \frac{\partial}{\partial R_i}$$

$$d^d R = \det \underset{\sim}{\Lambda}\, d^d x$$

$$\partial_\alpha = \frac{\partial}{\partial x_\alpha} = \frac{\partial R_i}{\partial x_\alpha} \frac{\partial}{\partial R_i} = \Lambda_{i\alpha} \nabla_i$$

$$\boxed{\sigma^C_{ij} = \frac{1}{\det \underset{\sim}{\Lambda}} \sigma^I_{i\alpha} \Lambda^T_{\alpha j} = \frac{1}{\det \underset{\sim}{\Lambda}} \Lambda_{i\alpha} \sigma^{II}_{\alpha\beta} \Lambda^T_{\alpha j}}$$

$$\underset{\sim}{\sigma}^C = \frac{1}{\det \underset{\sim}{\Lambda}} \underset{\sim}{\Lambda}\, \underset{\sim}{\sigma}^{II}\, \underset{\sim}{\Lambda}^T$$

Symmetric as required

Coupling to other fields

We are often interested in the coupling of target-space vectors like an electric field or the nematic director to elastic strain. How is this done? The strain tensor $u_{\alpha\beta}$ is a scalar in the target space, and it can only couple to target-space scalars, not vectors.

Answer lies in the polar decomposition theorem

$$\underset{\sim}{\Lambda} = \underset{\sim}{\Lambda}(\underset{\sim}{\Lambda}^T\underset{\sim}{\Lambda})^{-1/2}(\underset{\sim}{\Lambda}^T\underset{\sim}{\Lambda})^{1/2} \equiv \underset{\sim}{\Theta}\underset{\sim}{M}^{1/2}$$

$$\underset{\sim}{M} = \underset{\sim}{\Lambda}^T\underset{\sim}{\Lambda} = (\underset{\sim}{\delta} + 2\underset{\sim}{u}); \quad \underset{\sim}{\Theta} = \underset{\sim}{\Lambda}\underset{\sim}{M}^{-1/2}$$

$$\underset{\sim}{O}\underset{\sim}{O}^T = \underset{\sim}{\Lambda}\underset{\sim}{M}^{-1/2}(\underset{\sim}{\Lambda}\underset{\sim}{M}^{-1/2})^T = \underset{\sim}{\Lambda}\underset{\sim}{M}^{-1/2}\underset{\sim}{M}^{-1/2}\underset{\sim}{\Lambda}^T = \underset{\sim}{\Lambda}(\underset{\sim}{\Lambda}^T\underset{\sim}{\Lambda})^{-1}\underset{\sim}{\Lambda}^T = \underset{\sim}{\delta}$$

$\underset{\sim}{M}$ is symmetric and depends on $\underset{\sim}{u}$ only.

$\underset{\sim}{O}$ is an orthogonal, unimodular rotation matrix

Target-reference conversion

The rotation matrix $\underset{\sim}{O}$ converts target-space vectors E_i to reference-space vectors $\tilde{E}_\alpha$ and vice-versa

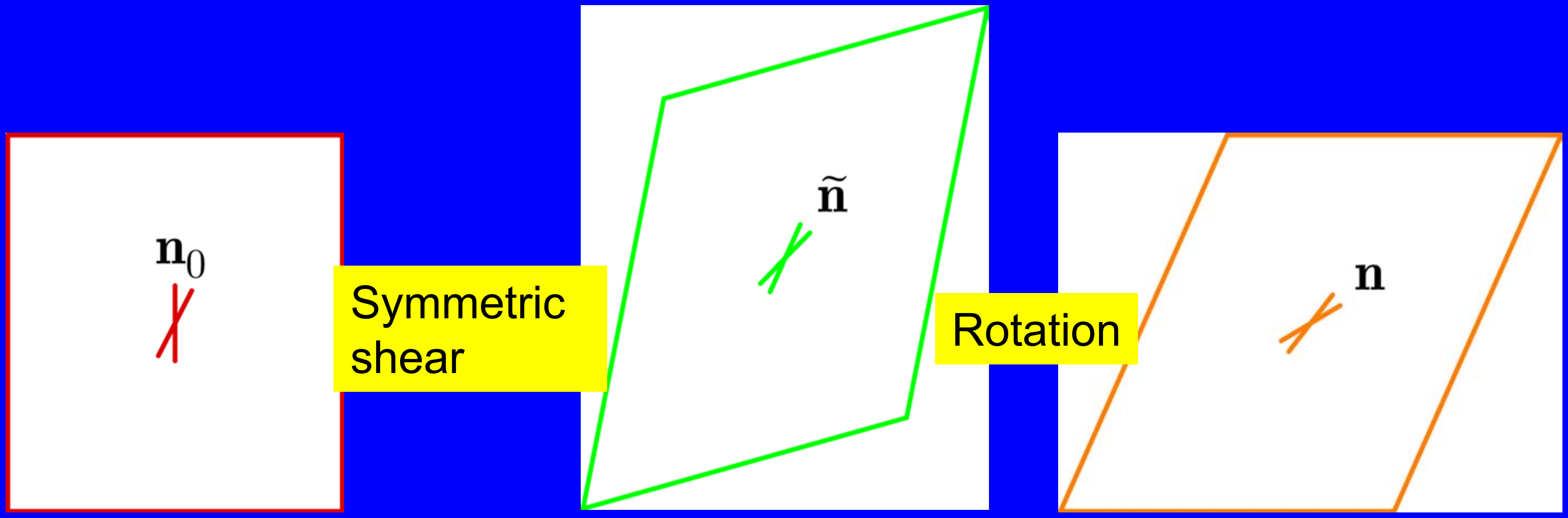
$$E_i = O_{i\alpha}\tilde{E}_\alpha; \quad \tilde{E}_\alpha = O^T_{\alpha i}E_i$$

If $\underset{\sim}{\Lambda}$ is symmetric, $\mathrm{O}_{i\alpha} = \delta_{i\alpha}$.

$$\begin{aligned} O_{i\alpha} &\approx \delta_{i\alpha} + \tfrac{1}{2}(\partial_\alpha u_i - \partial_i u_\alpha) \\ &\approx \delta_{i\alpha} - \varepsilon_{i\alpha k}\Omega_k \end{aligned}$$

To linear order in $\boldsymbol{u}$, $O_{i\alpha}$ has a term proportional to the antisymmetric part of the strain matrix.

Strain and Rotation



$\tilde{\mathrm{n}}$ is a reference space vector; it is equal to the target space vector that is obtained when $\underline{\Lambda}$ is symmetric

Sample couplings

Coupling of electric field to strain

$$u_{\alpha\beta}\tilde{E}_{\alpha}\tilde{E}_{\beta} = E_{i}O_{i\alpha}u_{\alpha\beta}O^{T}_{\beta j}E_{j} \equiv v_{ij}E_{i}E_{j}$$

$$\begin{aligned} \underset{\sim}{Q}\underset{\sim}{u}\underset{\sim}{Q}^{T} &= \tfrac{1}{2}\underset{\sim}{\Lambda}(\underset{\sim}{\Lambda}^{T}\underset{\sim}{\Lambda})^{-1/2}(\underset{\sim}{\Lambda}^{T}\underset{\sim}{\Lambda}-\underset{\sim}{\delta})(\underset{\sim}{\Lambda}^{T}\underset{\sim}{\Lambda})^{-1/2}\underset{\sim}{\Lambda}^{T} \\ &= \tfrac{1}{2}(\underset{\sim}{\Lambda}\underset{\sim}{\Lambda}^{T}-\underset{\sim}{\delta}) = \underset{\sim}{v} \end{aligned}$$

Free energy no longer depends on the strain $u_{\alpha\beta}$ only. The electric field defines a direction in the target space as it should

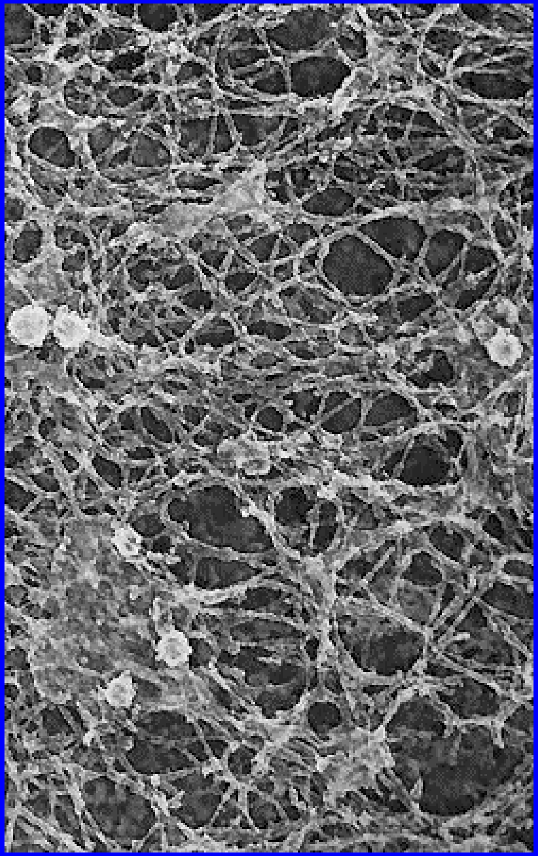
$$f^{T} = f(\underset{\sim}{u}) - gE_{i}E_{j}v_{ij}$$

$$\Lambda_{i\alpha} = \frac{\partial R_{i}}{\partial x_{\alpha}} = \frac{\partial R_{i}}{\partial x'_{\beta}}\frac{\partial x'_{\beta}}{\partial x_{\alpha}} = \Lambda'_{i\beta}\Lambda_{0\beta\alpha}$$

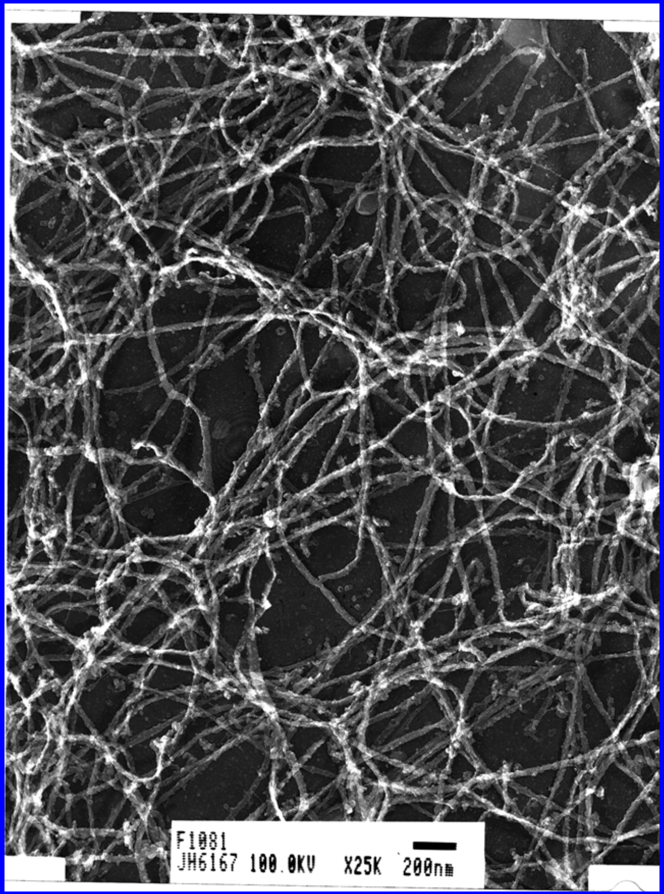
Energy depends on both symmetric and anti-symmetric parts of η'

$$\Lambda'_{i\alpha} = \delta_{i\alpha} + \eta'_{i\alpha}$$

Biopolymer Networks



cortical actin gel



neurofilament network

Characteristics of Networks

- Off Lattice
- Complex links, semi-flexible rather than random-walk polymers
- Locally randomly inhomogeneous and anisotropic but globally homogeneous and isotropic
- Complex frequency-dependent rheology
- Striking non-linear elasticity

Goals

- Strain Hardening (more resistance to deformation with increasing strain) – physiological importance
- Formalisms for treating nonlinear elasticity of random lattices
  - Affine approximation
  - Non-affine

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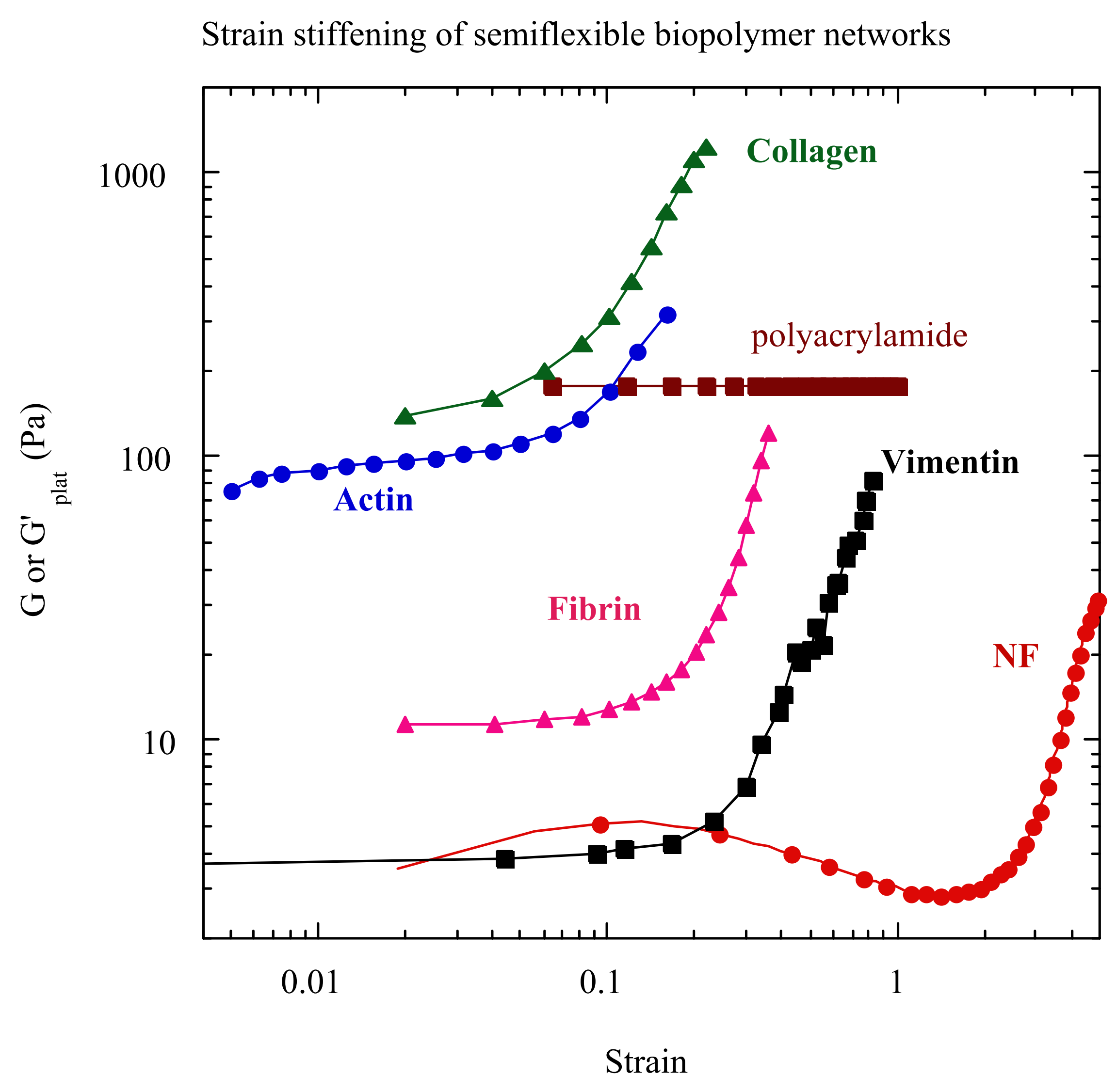
Different Networks



Max strain ~.25 except for vimentin and NF

Max stretch: L(Λ)/L~1.13 at 45 deg to normal

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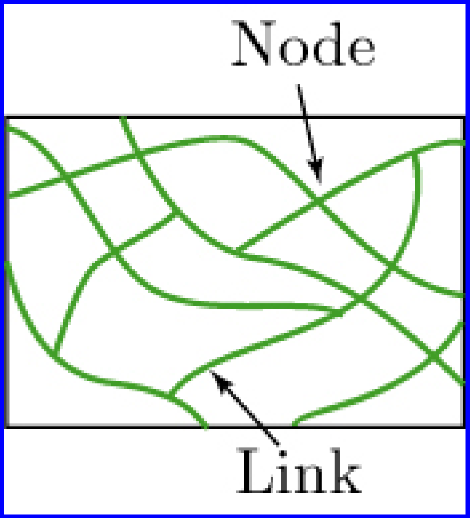
Semi-microscopic models



Random or periodic crosslinked network: Elastic energy resides in bonds (links or strands) connecting nodes

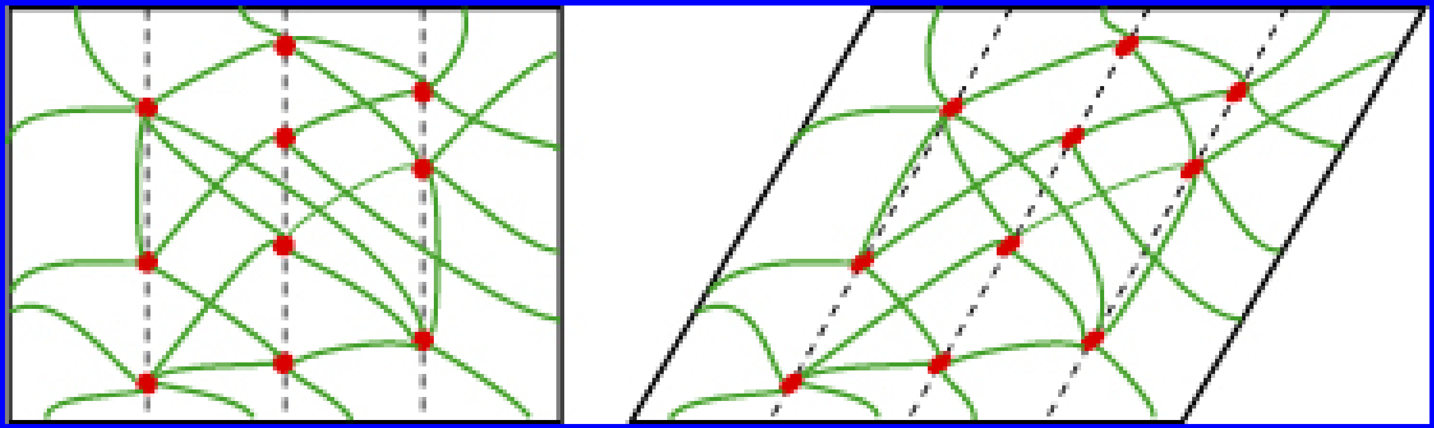
$\mathbf{R}_b$ = separation of nodes in bond b
$V_b(|\mathbf{R}_b|)$ = free energy of bond b

$$F = \sum_b V_b(\mathbf{R}_b) = N \langle V(\mathbf{R}) \rangle_{\mathbf{R}_0}$$

$$f = \frac{F}{V} = n_b \langle V(\mathbf{R}) \rangle_{\mathbf{R}_0}$$

n_b = Number of bonds per unit volume of reference lattice

Affine Transformations



Reference network:
Positions $\mathbf{R}_0$

$$f = \frac{F}{V} = n_b \left\langle V(\underset{\sim}{\Lambda}\mathbf{R}_0) \right\rangle_{\mathbf{R}_0}$$

Depends only on u_{ij}

Strained target network:
$R_i = \Lambda_{ij} R_{0j}$

$$\underset{\sim}{\Lambda} = \underset{\sim}{Q}(\underset{\sim}{\Lambda}^T \underset{\sim}{\Lambda})^{1/2} = \underset{\sim}{Q}(1 + 2\underset{\sim}{u})^{1/2}$$

$$\underset{\sim}{Q} = \underset{\sim}{\Lambda}(\underset{\sim}{\Lambda}^T \underset{\sim}{\Lambda})^{-1/2} : \text{ Orthogonal}$$

$$|\underset{\sim}{\Lambda}\mathbf{R}_b^0| = |(1 + 2\underset{\sim}{u})^{1/2}\mathbf{R}_b^0|$$

Example: Rubber

$$V(R) = \frac{3}{2} T \frac{R^2}{Nb^2}$$

Purely entropic force

$$F = n_b \left\langle V(\underset{\sim}{\Lambda}\mathbf{R}) \right\rangle_R = \frac{3}{2} \frac{T}{Nb^2} \left\langle \mathbf{R}_0 \underset{\sim}{\Lambda}^T \underset{\sim}{\Lambda} \mathbf{R}_0 \right\rangle_{R_0} = \frac{1}{2} T n_b \mathrm{Tr} \underset{\sim}{\Lambda}^T \underset{\sim}{\Lambda}$$

$$P(R) = \sqrt{\frac{3}{2\pi N b^2}} \exp\left[-\frac{3R^2}{2Nb^2} \right]$$

$$\left\langle R_{0i} R_{0j} \right\rangle = \frac{1}{3} \delta_{ij} N b^2$$

$$R_0^2 = Nb^2$$

Average is over the end-to-end separation in a random walk: random direction, Gaussian magnitude

Rubber : Incompressible Stretch

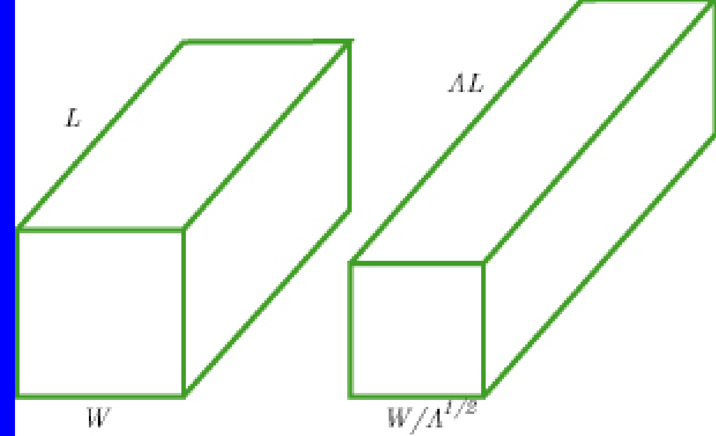
$$f = \frac{1}{2} T n_b \mathrm{Tr} \underset{\sim}{\Lambda}^T \underset{\sim}{\Lambda} = \frac{1}{2} T n_b \mathrm{Tr}(1 + 2\underset{\sim}{u})$$

Unstable: nonentropic forces between atoms needed to stabilize; Simply impose incompressibility constraint.

$$\mathbf{\Lambda} = \begin{pmatrix} \Lambda^{-1/2} & 0 & 0 \\ 0 & \Lambda^{-1/2} & 0 \\ 0 & 0 & \Lambda \end{pmatrix}$$

$$f = \frac{1}{2} n_b T \left(\Lambda^2 + \frac{2}{\Lambda} \right)$$

Rubber: stress -strain



$$f_z = \frac{\partial}{\partial L}(V f) = \frac{\partial (A_R L_R f)}{\partial \Lambda L_R} = A_R \frac{\partial f}{\partial \Lambda}$$

A_R= area in reference space

Engineering stress

$$\sigma^e = \frac{f_z}{A_R} = \frac{\partial f}{\partial \Lambda} = nT\left(\Lambda - \frac{1}{\Lambda^2}\right)$$

Physical Stress

$A = A_R/\Lambda$ = Area in target space

$$\sigma = \frac{f_z}{A} = \Lambda \frac{\partial f}{\partial \Lambda} = nT\left(\Lambda^2 - \frac{1}{\Lambda}\right)$$

Y=Young's modulus

$$Y = \frac{\sigma}{\gamma} = \frac{nT}{\gamma}\left((1+\gamma)^2 - \frac{1}{1+\gamma}\right) \sim 3nT$$

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General Case

Engineering stress: not symmetric

$$ {}^{\mathrm{e}}_{ij} = \frac{\partial f}{\partial \Lambda_{ij}} = n \left\langle V'(\underset{\sim}{\Lambda}\mathbf{R}_0) \frac{(\underset{\sim}{\Lambda}\mathbf{R}_0)_i}{|\underset{\sim}{\Lambda}\mathbf{R}_0|} R_{0j} \right\rangle_{\mathbf{R}_0} $$

$$ = n \left\langle \tau_i(\underset{\sim}{\Lambda}\mathbf{R}_0) R_{0j} \right\rangle = n \left\langle \tau(\underset{\sim}{\Lambda}\mathbf{R}_0) \frac{(\underset{\sim}{\Lambda}\mathbf{R}_0)_i}{|\underset{\sim}{\Lambda}\mathbf{R}_0|} R_{0j} \right\rangle $$

$$ \sigma_{ij} dS_j = \sigma^{\mathrm{e}}_{ij} dS^{\mathrm{ref}}_j $$

$$ dS_i = \det \mathbf{\Lambda}\; \Lambda^{-1}_{ji} dS^{\mathrm{ref}}_j $$

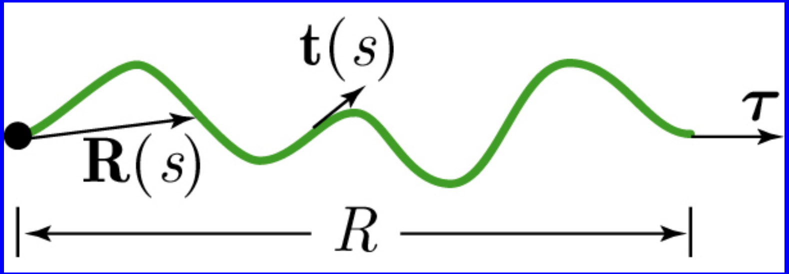
Central force

$$ \tau(R) = \frac{dV(R)}{dR} $$

Physical Cauchy Stress: Symmetric

$$ \sigma_{ij} = \frac{n}{\det \underset{\sim}{\Lambda}} \left\langle \frac{\tau(\underset{\sim}{\Lambda}\mathbf{R}_0)}{|\underset{\sim}{\Lambda}\mathbf{R}_0|} \Lambda_{ik} R_{0k} \Lambda_{jl} R_{0l} \right\rangle_{\mathbf{R}_0} $$

Semi-flexible Stretchable Link



$$R[\mathbf{t}, v] \equiv L = \int_0^{L_0} ds \frac{dR_z}{ds}$$
$$\approx \int_0^{L_0} ds\, v\left[1 - \tfrac{1}{2} \mid \mathbf{t}_\perp \mid^2\right]$$

$$\mid \mathbf{t}(s) \mid = 1; \qquad \mathbf{t}(s) = \left(\mathbf{t}_\perp(s), \sqrt{1 - \mid \mathbf{t}_\perp(s) \mid^2}\right)$$

$$\frac{d\mathbf{R}}{ds} = v(s)\mathbf{t}(s)$$

$$\left|\frac{d\mathbf{R}}{ds}\right| = v$$

t = unit tangent
v = stretch

$$H = \frac{1}{2}\int ds \left[\kappa\left(\frac{d\mathbf{t}_\perp}{ds}\right)^2 + v\tau \mid \mathbf{t}_\perp \mid^2 + K(v-1)^2\right]$$

Length-force expressions

$L(\tau, K)$ = equilibrium length at given τ and K

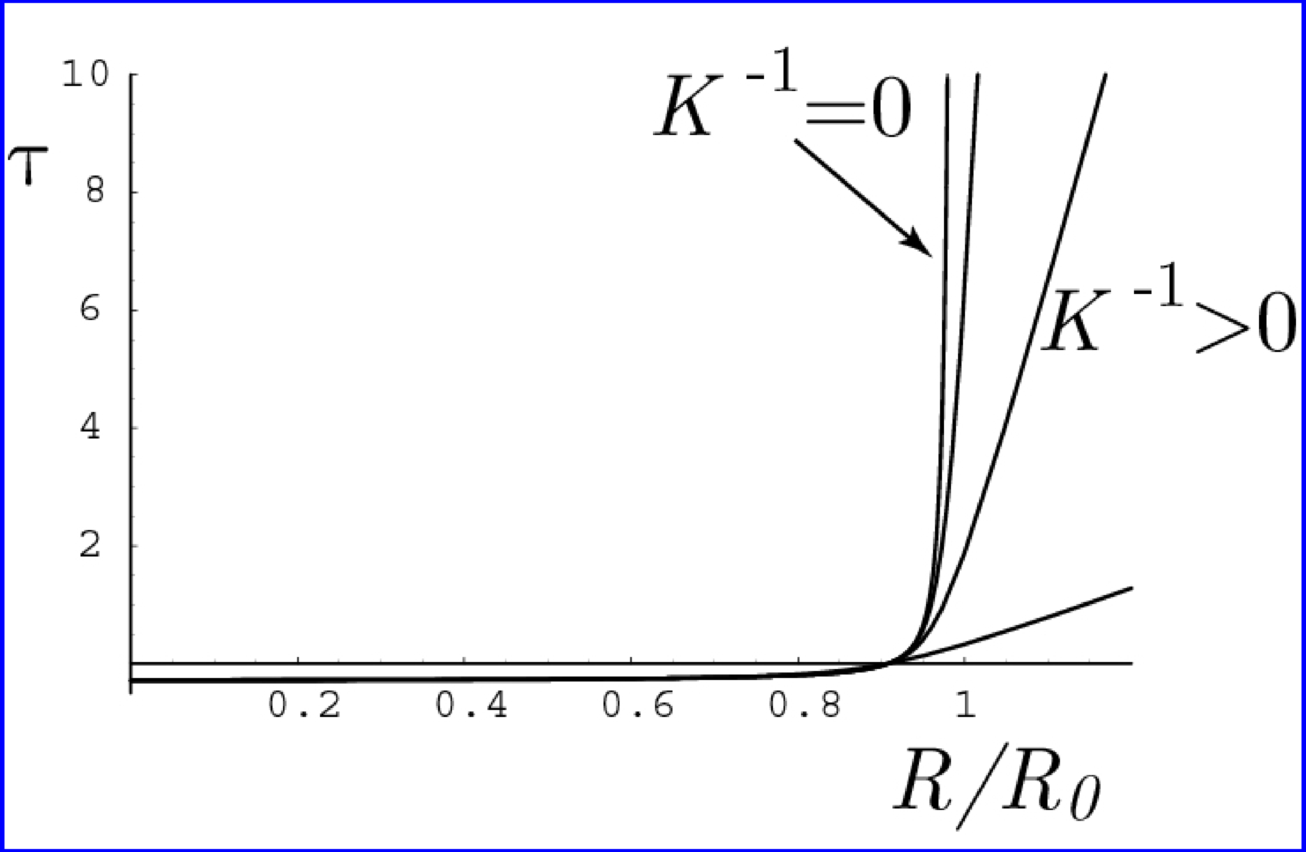
$$L(\tau, K) = \left(1 + \frac{\tau}{K}\right) L_0 [1 - g(\varphi(\tau, K))];$$

$$g(\varphi) = \tfrac{1}{2} \left\langle |\, \mathbf{t}_\perp \,|^2 \right\rangle = \frac{1}{\pi^2} \frac{L_0}{L_p} \sum_{n=1}^{\infty} \frac{1}{n^2 + \varphi};$$

$$= \frac{L_0}{L_p} \frac{\pi\sqrt{\varphi}\coth(\pi\sqrt{\varphi}) - 1}{\pi^2 \varphi}$$

$$\varphi(\tau, K) = \frac{L_0^2}{\kappa \pi^2} \tau \left(1 + \frac{\tau}{K}\right); \; L_p = \frac{\kappa}{k_B T}$$

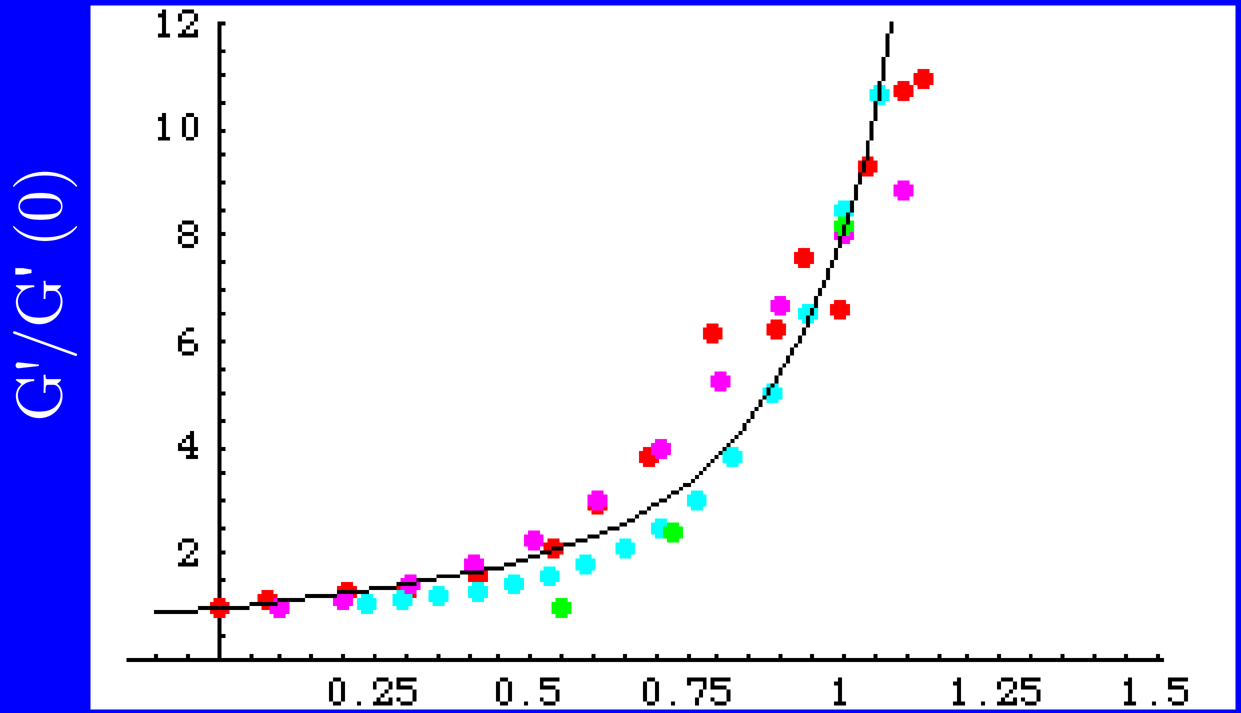
Force-extension Curves



Scaling at "Small" Strain

Theoretical curve: calculated from $K^{-1}=0$

zero parameter fit to everything



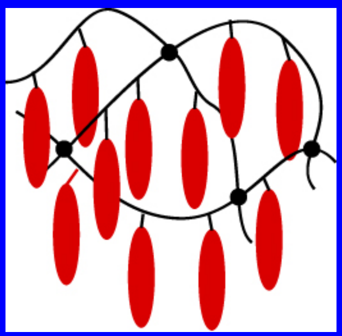
Strain/strain8

What are Nematic Gels?

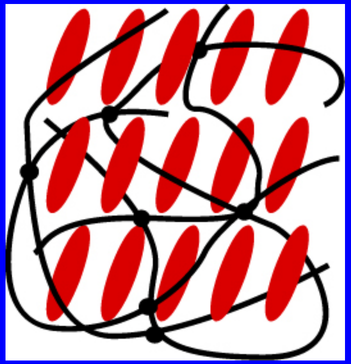
- Homogeneous Elastic media with broken rotational symmetry (uniaxial, biaxial)
- Most interesting - systems with broken symmetry that develops spontaneously from a homogeneous, isotropic elastic state

Examples of LC Gels

1. Liquid Crystal Elastomers - Weakly crosslinked liquid crystal polymers

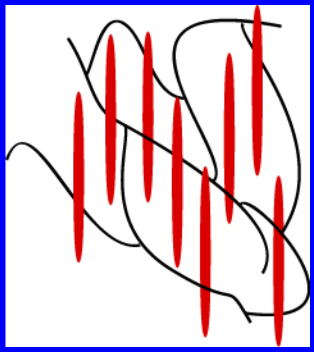


Nematic



Smectic-*C*

2. Tanaka gels with hard-rod dispersion

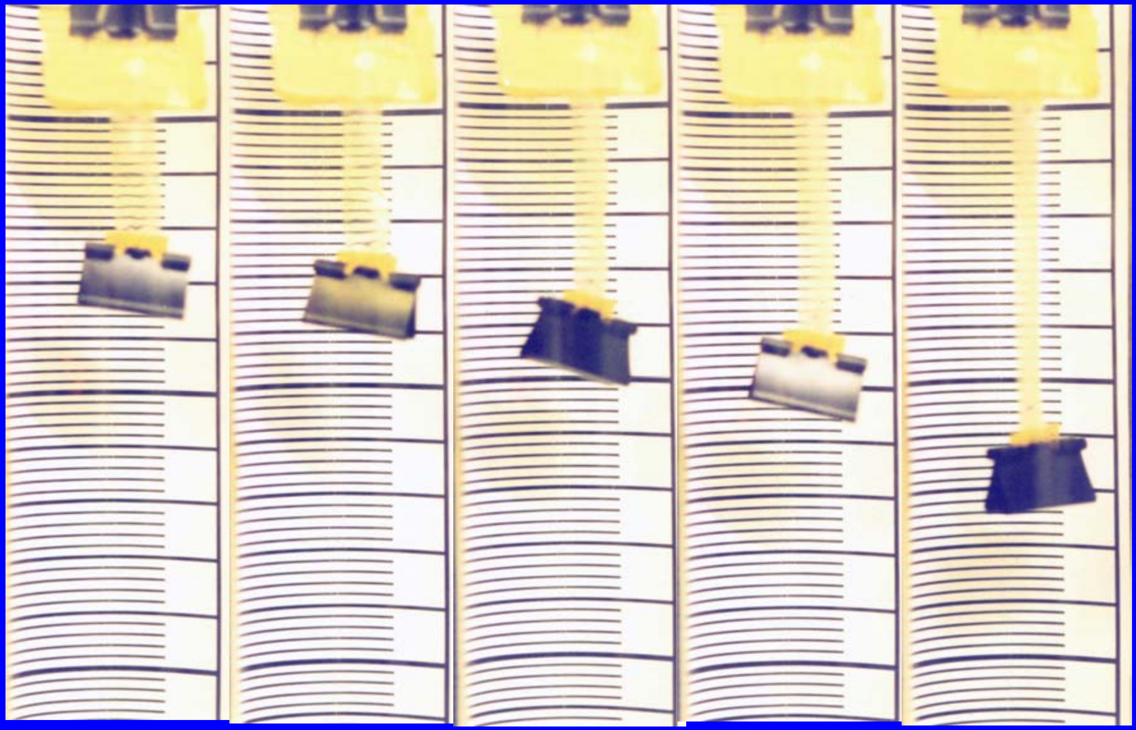


3. Anisotropic membranes

4. Glasses with orientational order

Properties I

- **Large thermoelastic effects** - Large thermally induced strains - artificial muscles

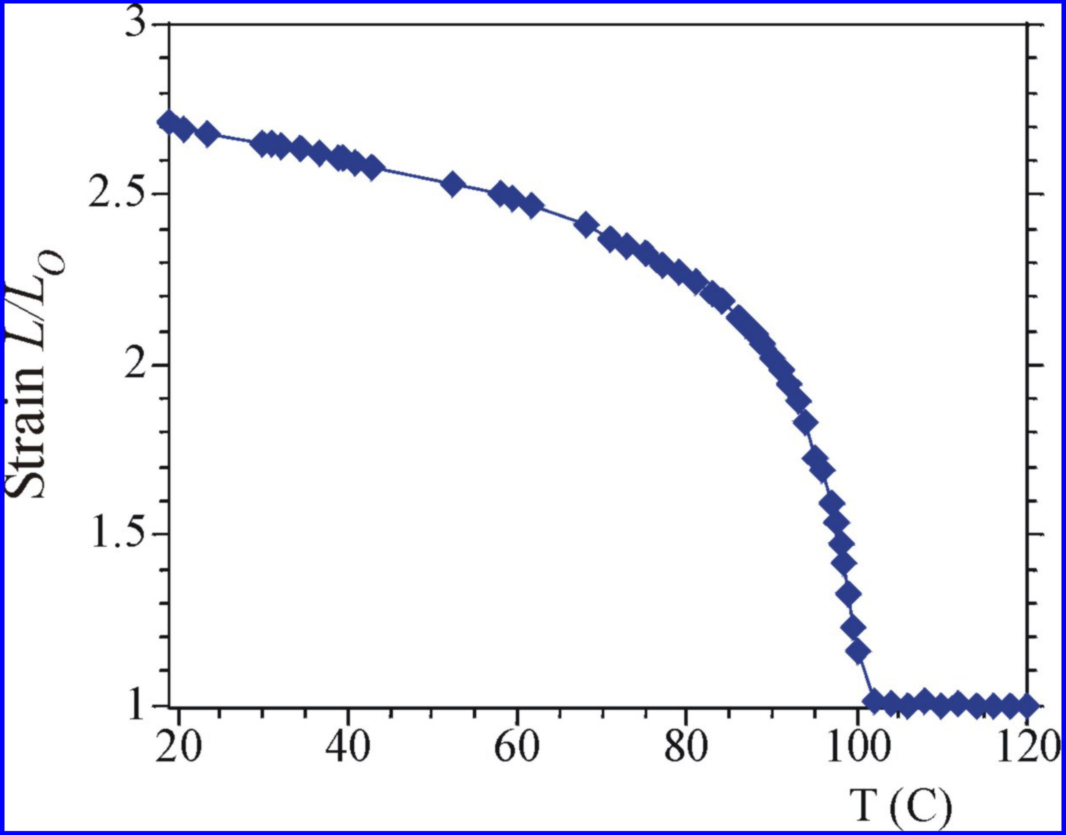


Courtesy of
Eugene Terentjev

300% strain

Properties II

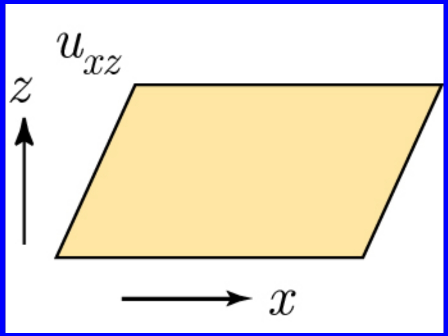
Large strain in small temperature range



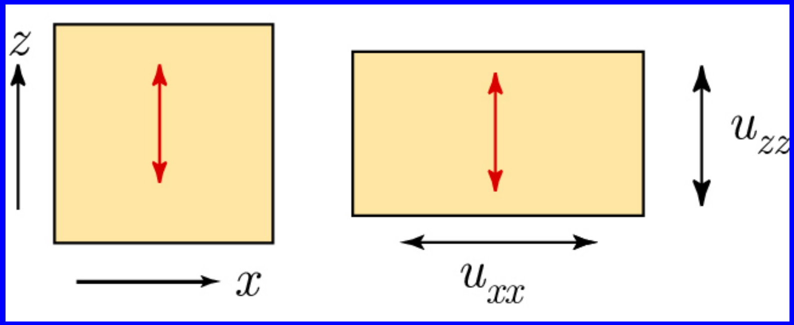
Terentjev

Properties III

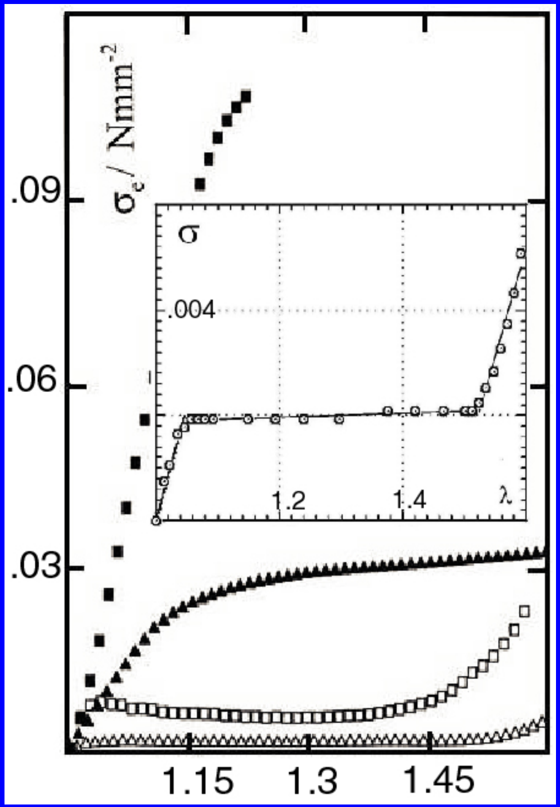
- Soft or "Semi-soft" elasticity



Vanishing xz shear modulus



Soft stress-strain for stress perpendicular to order



Warner Finkelmann

Model for Isotropic-Nematic trans.

$$f = \frac{1}{2} B u_{\alpha\alpha}^2 + \mu \mathrm{Tr}\underline{\tilde{u}}^2 - C\mathrm{Tr}\underline{\tilde{u}}^3 + D\left(\mathrm{Tr}\underline{\tilde{u}}^2\right)^2$$

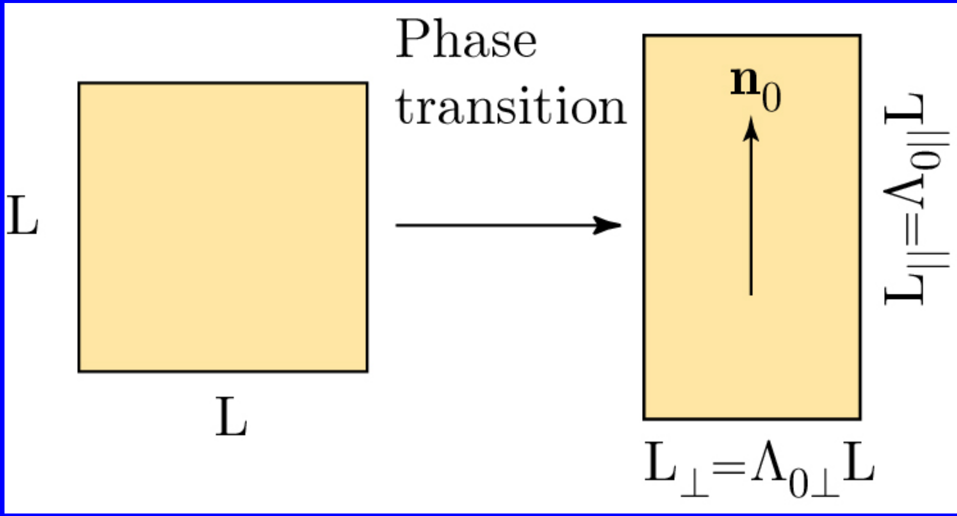
$$\tilde{u}_{\alpha\beta} = u_{\alpha\beta} - \frac{1}{3}\delta_{\alpha\beta} u_{\gamma\gamma}$$

μ approaches zero signals a transition to a nematic state with a nonvanishing

$$\tilde{u}_{\alpha\beta} = S\left(n_\alpha n_\beta - \frac{1}{3}\delta_{\alpha\beta}\right)$$

Spontaneous Symmetry Breaking

Phase transition to anisotropic state as μ goes to zero



$$\underline{u}_0 = \tfrac{1}{2}\left(\underline{\Lambda}_0^T \underline{\Lambda}_0 - \underline{\delta}\right)$$

$$\underline{\Lambda}_0 = \sqrt{\underline{\delta} + 2\underline{u}_0}$$

Direction of $\mathbf{n}_0$ is arbitrary

$$\tilde{u}_{\alpha\beta} = \tilde{u}_{0\alpha\beta} = \Psi\left(n_\alpha^0 n_\beta^0 - \tfrac{1}{3}\delta_{\alpha\beta}\right)$$

Symmetric-Traceless part

$$u_{\alpha\alpha} \sim \Psi^2$$

Strain of New Phase

$$R_i(\mathbf{x}) = \Lambda_{0ij} x_j + \delta u_i(\mathbf{x})$$
$$= x'_i + u'_i(\mathbf{x}')$$

$\underline{u}'$ is the strain relative to the new state at points x'

$$\Lambda_{ij} = \frac{\partial R_i}{\partial x_j} = \frac{\partial R_i}{\partial x'_k}\frac{\partial x'_k}{\partial x_j} = \Lambda'_{ik}\Lambda_{0kj}$$

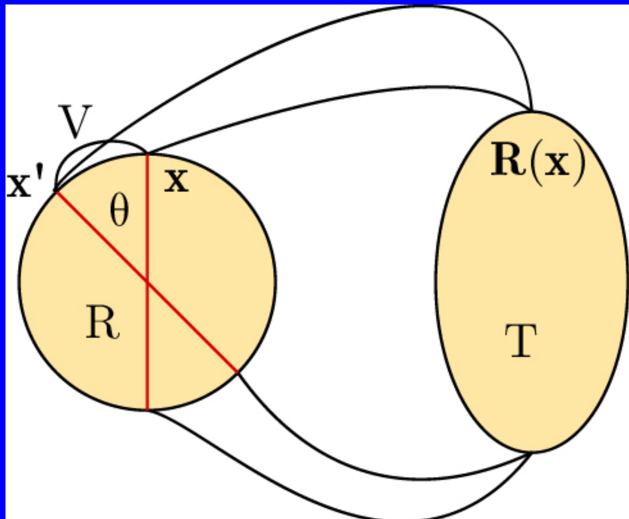
$$\delta \underset{\sim}{u} = \underset{\sim}{u} - \underset{\sim}{u}_0$$
$$= \tfrac{1}{2}\left(\underset{\sim}{\Lambda}^T \underset{\sim}{\Lambda} - \underset{\sim}{\Lambda}_0^T \underset{\sim}{\Lambda}_0\right)$$
$$= \underset{\sim}{\Lambda}_0^T \underset{\sim}{u}' \underset{\sim}{\Lambda}_0$$

$\delta\underline{u}$ is the deviation of the strain relative to the original reference frame R from $\underline{u}_0$

$$\underline{u}' = \tfrac{1}{2}\left(\underline{\Lambda}'^T \underline{\Lambda}' - \underline{\delta}\right) \approx \tfrac{1}{2}\left(\underset{\sim}{\eta}' + \underset{\sim}{\eta}'^T\right)$$

$\delta\underline{u}$ is linearly proportional to $\underline{u}'$

Elasticity of New Phase



Rotation of anisotropy direction costs no energy

$$r = \frac{\Lambda_{0||}^2}{\Lambda_{0\perp}^2}$$

$$\underline{u}' = (\underline{\Lambda}_0^T)^{-1}\left(\underline{V}\,\underline{u}_0\underline{V}^{-1} - \underline{u}_0\right)\underline{\Lambda}_0^{-1}$$

$$= \tfrac{1}{4}(r-1)\begin{pmatrix} 1-\cos 2\theta & \frac{1}{\sqrt{r}}\sin 2\theta \\ \frac{1}{\sqrt{r}}\sin 2\theta & -\frac{1}{r}(1-\cos 2\theta) \end{pmatrix}$$

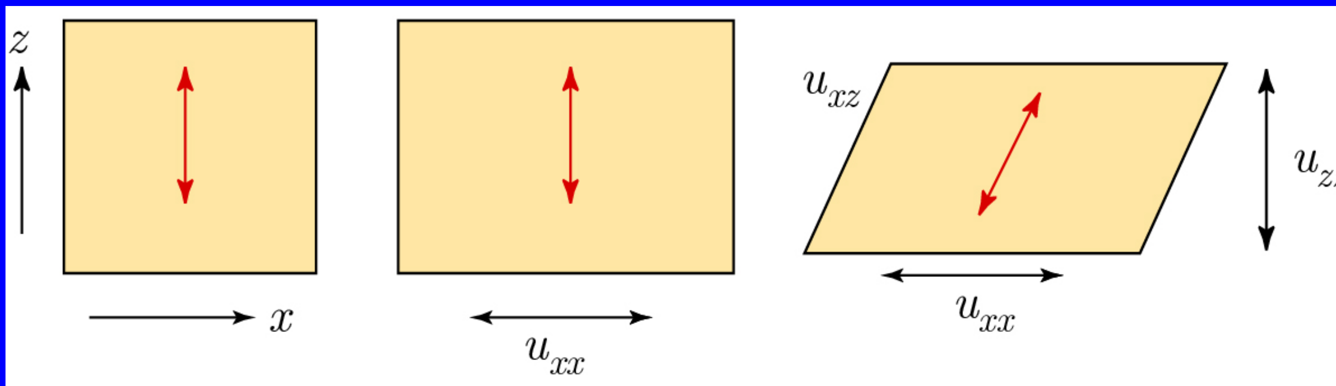
$$\boxed{u'_{xz} \sim \frac{(r-1)}{4\sqrt{r}}\theta}$$

$C_5=0$ because of rotational invariance

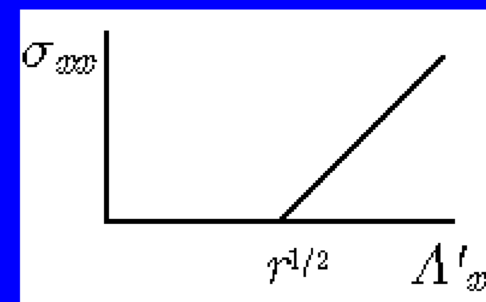
$$f_{\mathrm{el}} = \tfrac{1}{2}C_1 u'^2_{zz} + C_2 u'_{zz}u'_{\nu\nu} + \tfrac{1}{2}C_3 u'_{\nu\nu}u'_{\nu\nu} + C_4 u'_{\nu\tau}u'_{\nu\tau} + \cancel{C_5 u'_{z\nu}u'_{z\nu}}$$

This 2nd order expansion is invariant under all *U* but only infinitesimal *V*

Soft Extensional Elasticity



$$\underline{u} = \frac{1}{4}(r-1)\begin{pmatrix} 1-\cos 2\theta & \frac{1}{\sqrt{r}}\sin 2\theta \\ \frac{1}{\sqrt{r}}\sin 2\theta & -\frac{1}{r}(1-\cos 2\theta) \end{pmatrix}$$



$$u_{zz} = -\frac{1}{r}u_{xx}$$

$$u_{xz} = \frac{1}{\sqrt{2r}}\sqrt{u_{xx}(r-1-2u_{xx})}$$

Strain u_{xx} can be converted to a zero energy rotation by developing strains u_{zz} and u_{xz} until u_{xx} =(r-1)/2

Frozen anisotropy: Semi-soft

System is now uniaxial – why not simply use uniaxial elastic energy? This predicts linear stress-stain curve and misses lowering of energy by reorientation:

$$f = \tfrac{1}{2} C_1 u_{zz}^2 + C_2 u_{zz} u_{\nu\nu} + \tfrac{1}{2} C_3 u_{\nu\nu}^2 + C_4 u_{\nu\tau}^2 + C_5 u_{\nu z}^2$$

Model Uniaxial system: Produces harmonic uniaxial energy for small strain but has nonlinear terms – reduces to isotropic when $h=0$

$$f^h(\underline{u}) = f(\underline{u}) - h u_{zz}$$

$f(u)$: isotropic

Rotation $$u \rightarrow u' = u + \theta \begin{pmatrix} -2u_{xz} & u_{xx} - u_{zz} \\ u_{xx} - u_{zz} & 2u_{xz} \end{pmatrix}$$

$$f^h(\underline{u}') = f(\underline{u}) - h(u_{zz} + 2\theta u_{xz})$$

Semi-soft stress-strain

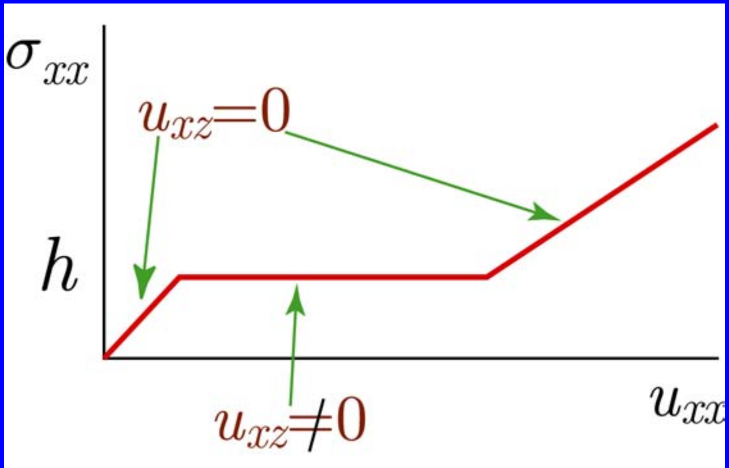
Ward Identity

$$\frac{df^h}{d\theta} = -2hu_{xz} = 2\sigma_{xz}(u_{xx} - u_{zz}) + 2(\sigma_{zz} - \sigma_{xx})u_{xz}$$

$$\sigma_{xz} = \frac{(\sigma_{xx} - h)u_{xz}}{u_{xx} - u_{zz}} \Rightarrow u_{xz} = 0 \text{ or } \sigma_{xx} = h$$

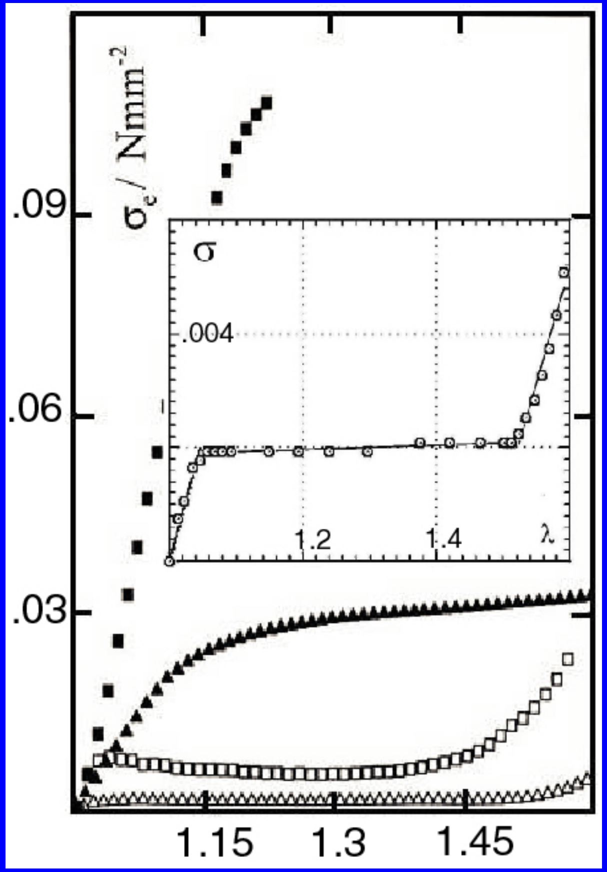
$$\sigma_{\alpha\beta} = \frac{\partial f^h}{\partial u_{\alpha\beta}}$$

Second Piola-Kirchoff stress tensor.

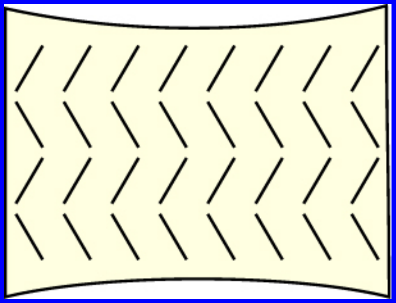


Semi-soft Extensions

Break rotational symmetry



Stripes form in real systems: semi-soft, BC



Not perfectly soft because of residual anisotropy arising from crosslinking in the the nematic phase - **semi-soft.** length of plateau depends on magnitude of spontaneous anisotropy r.

Warner-Terentjev

Note: Semi-softness only visible in nonlinear properties

Finkelmann, et al., J. Phys. II **7**, 1059 (1997);
Warner, J. Mech. Phys. Solids **47**, 1355 (1999)

Softness with Director

N_α = unit vector along uniaxial direction in reference space; layer normal in a locked SmA phase

$$\tilde{\mathbf{n}}_\alpha = (\tilde{n}_\nu, \tilde{n}_z)$$

$$\tilde{n}_\nu^2 = 1 - (N_\alpha \cdot \tilde{n}_\alpha)^2 \equiv c_\nu^2; \qquad u_{zz} = N_\alpha u_{\alpha\beta} N_\beta, \text{ etc.}$$

Red: SmA-SmC transition

$$\begin{aligned} f &= \tfrac{1}{2} C_1 u_{zz}^2 + C_2 u_{zz} u_{\nu\nu} + \tfrac{1}{2} C_3 u_{\nu\nu}^2 + C_4 u_{\nu\tau}^2 + \lambda_1 \tilde{n}_\nu^2 u_{zz} \\ &\quad + C_5 u_{\nu z}^2 + D_2 \tilde{n}_\nu \tilde{n}_z u_{\nu z} + \tfrac{1}{2} D_1 \tilde{n}_\nu^2 + \tfrac{1}{4} g \tilde{n}_\nu^4 + \lambda_2 \tilde{n}_\nu^2 u_{\tau\tau} + \cdots \\ &= \tfrac{1}{2} C_1 u_{zz}^2 + C_2 u_{zz} u_{\nu\nu} + \tfrac{1}{2} C_3 u_{\nu\nu}^2 + C_4 u_{\nu\tau}^2 \\ &\quad + \tfrac{1}{2} D_1 [\tilde{n}_\nu + (D_2 / D_1) u_{\nu z}]^2 + [C_5 - \tfrac{1}{2}(D_2^2 / D_1)] u_{\nu z}^2 \end{aligned}$$

Director relaxes to zero

$$C_5^R = C_5 - \frac{1}{2}\frac{D_2^2}{D_1} = 0 \Rightarrow \text{Soft}$$