I'd love to hear if anyone has ideas on how to solve these; please email me with questions or comments: chb at sas dot upenn dot edu.

1, Discriminant. June 2008, updated 20081110

Show that the discriminant of $P(u) = (u^n + K)^2 - u^{2n-1}(T + u)^2$ is:

$$(-1)^n K^{4(n-1)}(K + (-T)^n)^2 Q_{2n+1}(K, T) \tag{1}$$

where $Q_{2n+1}(K, T)$ is a polynomial of degree $2n + 1$ in T. Find an expression for Q_{2n+1} if possible.

This polynomial arises in the classification of radially anisotropic Delaunay (RAD) surfaces[1]. These are extremizers of the following functional:

$$2\pi\gamma \int drdz \left[r\sqrt{1 + \dot{r}^2} \left(1 + \frac{K}{r^{2n}} \right) - \frac{1}{2}r^2 \right]$$

T is the first integral of this functional, a "tension". The discriminant Q characterizes the boundaries of the different "phases" where we get particular families of surfaces.

We may interpret the factors other than Q_{2n+1} in the following way. First, the factors of K out front correspond to a $K = 0$ solution - this corresponds to the ordinary Delaunay surface case, and the "multiple roots" are just degenerate solutions $u = 0$, as may be confirmed by letting $K = 0$ in $P(u)$, and we will discard these.

The next factor $(K + (-T)^n)^2$ corresponds to setting $K = -(-T)^n$. In this case, $P(u)$ factors easily:

$$P(u) = (u^n + (-T)^n)^2 - (T + u)^2 u^{2n-1} \tag{2}$$

$$= (T + u)^2 [\textstyle\sum_{j=0}^{2n} (-1)^j T^j u^{2n-j} - u^{2n-1}] \tag{3}$$

The fact that $u^n + (-T)^n$ has a factor of $T + u$ follows easily from polynomial division and probably more sophisticated arguments. Roughly speaking it generalizes the factorization of the difference of squares. What this shows is that the case $K = -(-T)^n$ corresponds to a double root at $u = -T$, and since $T = -(-K)^{1/n} > 0$ when $K > 0$, this double root is at a negative real value of u and so does not correspond to a surface when $K > 0$.

This reduces the study of the number of positive real solutions of $P(u)$ when $K > 0$ to the study of the polynomial Q_{2n+1}.

I suspect that Q_{2n+1} has only one real root in general (bonus points if you can show this), which would mean that RAD surfaces cannot form unduloids after K increases past a certain critical K_c for any n, where K_c woud be the root of the discriminant of Q_{2n+1}.

2, Elliptic Integrals and Circles. July 2008

Here are a few more problems from the RAD surface paper[1]. Below, T is a "tension", K is roughly an elastic constant, and $z_j(r)$ is the generating curve for an axisymmetric surface which minimizes the functional (note that we've set $n = 1$):

$$2\pi\gamma \int drdz \left[r\sqrt{1+\dot{r}^2}\left(1+\frac{K}{r^2}\right) - \frac{1}{2}r^2 \right]$$

Given:

$$\begin{aligned} P(u) &= (u+K)^2 - u(T+u)^2 \\ &= -u^3 + (1-2T)u^2 + (2K-T^2)u + K^2 \end{aligned}$$

with u_1, u_2, u_3 its roots, with $u_1 < u_2 < u_3$ when all are real. The following asymptotic problems on the integral $z_j(r) = \int_{r^2}^{u_j} \frac{(T+u)du}{2\sqrt{P(u)}}$ occur as u_j becomes quite separated from one (or two) of the other roots. In both problems below, we want proofs of the numerical fact that $z_j(r)$ approaches a semicircular (or perhaps elliptical) function $A\sqrt{R_1^2 - (r-R_2)^2}$.

We can show that $u_1 \sim \frac{K^2}{T^2-2K}$, $u_{2,3} \sim -T \pm \sqrt{2K - T - \frac{K^2}{T}}$ as $K, T \to \infty$, by dominant balance arguments. For u_1 balance the terms K^2 and $(2K - T^2)u$, and for $u_{2,3}$ solve the following "quadratic equation":

$$\begin{aligned} P(u) &= u(-u^2 + (1-2T)u + (2K-T^2) + K^2/u) = 0 \\ u &\sim \frac{(2T-1) \pm \sqrt{(1-2T)^2 - 4(-1)(2K-T^2+K^2/u)}}{2(-1)} \\ &= -T + \frac{1}{2} \pm \sqrt{2K - T + \frac{1}{4} + \frac{K^2}{u}} \\ &\sim -T + \frac{1}{2} \pm \sqrt{2K - T + \frac{1}{4} + \frac{K^2}{-T}} \\ &\sim -T \pm \sqrt{2K - T - \frac{K^2}{T}} \end{aligned}$$

a, Tiny bubble as $|T| \to \infty$, Updated 20081110

When $|T| \to \infty$ at constant K, the above expressions for $u_1, u_{2,3}$ simplify:

$$\begin{aligned} u_1 &\sim \frac{K^2}{T^2 - 2K} \sim \frac{K^2}{T^2} \\ u_{2,3} &\sim -T \pm \sqrt{-T + 2K} \sim -T \pm \sqrt{-T} \end{aligned}$$

Note that this must break down wherever u_1 collides with u_2. This gives us a Cauchy-like "distance to the nearest singularity" which controls the convergence of these asymptotics.

We can find these points by computing the roots of $Q_3(T,K) = 4K+27K^2-18KT-T^2+4T^3$, the cubic factor of the discriminant of $P(u)$, as the discriminant gives the condition for double roots. It turns out that if we name the roots $T_1 < 0 < T_2 < T_3$ when $0 < K < 1/27$, or $T_1 < 0$ and T_2 and T_3 are the complex conjugate pair when $K > 1/27$, at T_1 and T_2 (if T_2 is real), u_1 and u_2 collide. Hence below, we only can take asymptotics when, $|T| > \max(|T_1|, |T_2|)$.

We find numerically and can show that as $|T| \to \infty$:

$$
\begin{aligned}
z_1(r) &= \int_{r^2}^{u_1} \frac{(T+u)du}{2\sqrt{P(u)}} \\
&\sim -\frac{T}{|T|}\sqrt{u_1 - r^2}
\end{aligned}
$$

As my coauthor points out, there was a typo with a factor of $\frac{1}{2}$ missing in the first line, hence the following heuristic argument actually does work, contrary to my mistake earlier:

$$
\begin{aligned}
z_1(r) &\sim \int_{r^2}^{u_1} \frac{(T+u)du}{2\sqrt{(u_1-u)((T+u)^2+T)}} \\
&\sim \int_{r^2}^{u_1} \frac{T du}{2\sqrt{(u_1-u)}|T|} \\
&\sim -\frac{T}{|T|}\sqrt{u_1 - r^2}
\end{aligned}
$$

So the open part of the problem is to improve these asymptotics in order to show the empirical fact that $z_1(r)$ is actually well-described by an ellipse for a larger range of $|T|$. This ellipse of course must circular in the large $|T|$ limit.

Hence the adjustment will be to find some function $f(T)$ in $z_1(r) \sim -\frac{T}{|T|}f(T)\sqrt{u_1 - r^2}$ where $f(T)$ approaches 1 as T increases. It can be shown from the integral that $|z_1(0)|$ diverges as $T \to T_1, T_2$. Hence, $f(T)$ should diverge at T_1 or T_2, and decrease to 1 as $|T| \to \infty$. Numerically, $f(T)$ diverges like $\log(T - T_1)$ near T_1 and behaves like $1 + K/T^2$ as $T \to \infty$.

It would be nice to see all of this in greater generality.

b, Torus as $K \to \infty$

Show that if we define T_{torus} to be the value of T such that:

$$
\begin{aligned}
0 &= \int_{u_2}^{u_3} \frac{(T+u)du}{\sqrt{P(u)}} \\
&= \frac{T+u}{\sqrt{u_3-u_1}} K\left(\sqrt{\frac{u_3-u_1}{u_3-u_2}}\right) + \sqrt{u_3-u_1} E\left(\sqrt{\frac{u_3-u_1}{u_3-u_2}}\right)
\end{aligned}
$$

then $T_{torus} \sim -3K - \frac{2}{3}$ as $K \to \infty$, and hence $u_1 \sim 1/9$, $u_{2,3} \sim 3K \pm \sqrt{\frac{16}{3}K}$.

Next show that the elliptic integral below has the following asymptotic behavior as $K \to \infty$ (with $T = T_{torus}$), and $u_2 < r^2 < u_3$:

$$z_3(r) = \int_{r^2}^{u_3} \frac{(T_{torus} + u)du}{2\sqrt{P(u)}}$$
$$\sim \sqrt{\frac{4}{9} - (r - \sqrt{3K})^2}$$

The scales $4/9 = (2/3)^2$ and $\sqrt{3K}$ can be derived from our values of u_2, u_3 via $\frac{2}{3} \sim \frac{\sqrt{u_3} - \sqrt{u_2}}{2}$ and $\sqrt{3K} \sim \frac{\sqrt{u_3} + \sqrt{u_2}}{2}$.

References

[1] B. G. Chen and R.D. Kamien, http://arxiv.org/abs/0811.2193 (2008).