Model 2 solved (conclusion)

$$H_I = f(t)\left[g\int d^3x\phi(\vec{x},t)\rho(\vec{x}) - a\right]$$

a is the vacuum energy counterterm chosen so that

$$\langle 0|S|0\rangle = 1$$

We have already argued that as $T\to\infty$, $a\to E_0$, where E_0 is the vacuum energy of the interacting theory (without the counterterm a). Finding a is going to give us E_0. Now it is clear what the addition of a constant to the Hamiltonian does to $U_I(\infty,-\infty)$. The constant just exponentiates. Let's see this come out of our diagrammatic perturbation theory.

There are now three connected diagrams

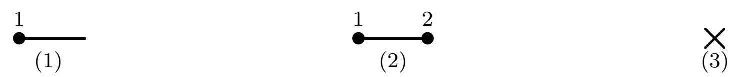

Diagram (3), which has no lines coming out, is for the counterterm.

$$S = U_I(\infty,-\infty) =: e^{(1)+(2)+(3)} := e^{(2)+(3)} : e^{(1)} :$$

(Since (2) and (3) are just numbers.) To set $\langle 0|S|0\rangle = 1$ is to set

$$e^{(2)+(3)} = 1 \quad \text{i.e.} \quad (3) = -(2)$$

Since a is an addition to the Hamiltonian, it had better be purely real. Diagram (3) is then pure imaginary, and diagram (2) in order to be cancellable had better come out pure imaginary. It didn't come out pure imaginary in Model (1), but there the source was time dependent.

Photons don't scatter off nailed down charges. Mesons don't scatter off nailed down nucleons. They only scatter off real nucleons (or off nailed down nucleons if there is some dynamical charged field in the theory).

$$(1) = -ig\int f(t)\rho(x)\phi(x)d^3x dt$$

$$\text{and}\quad \phi(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}}\left(e^{-ik\cdot x}a_{\vec{k}} + e^{ik\cdot x}a^\dagger_{\vec{k}}\right) \quad \text{so}$$

$$(1) = -ig\int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega_{\vec{k}}}}\left(\tilde{f}(\omega_{\vec{k}})a^\dagger_{\vec{k}}\tilde{\rho}(\vec{k}) + \overbrace{\tilde{f}(-\omega_{\vec{k}})}^{\tilde{f}(\omega_{\vec{k}})^*}a_{\vec{k}}\underbrace{\tilde{\rho}(-\vec{k})}_{\tilde{\rho}(\vec{k})^*}\right)$$

(hermitian conjugates)

$f(t)$, our turning on and off function, has a Fourier transform that looks like

As $T\to\infty$, $\tilde{f}(\omega_{\vec{k}})$ goes to zero for every $\omega_{\vec{k}}\neq 0$ and since $\omega_{\vec{k}}\geq\mu$ $\tilde{f}(\omega_{\vec{k}})\to 0$ for all $\vec{k}$. That is (1) $\to 0$ as $T\to\infty$. (as long as we can set $(3) = -(2)$) we have found

$$\boxed{S = 1}$$

This theory is a complete washout as far as scattering is concerned. While this was easy to see in the formalism we have built up, it was not easy when they were evaluating the theory in the Born approximation. Not until miraculous cancellations of all the terms at 4th order in the Born series occurred did people realize that they should try to prove $S = 1$ to all orders.

Why is $S = 1$? A time independent source can impart no energy. Since it can only create mesons one at a time and since $\omega = 0$ is not on the mass shell, it cannot create mesons.

This result holds in the massless theory too. Since there is clearly no scattering for all $\vec{k} \neq 0$, you only have to prove that for wave packets centered about $\vec{k} = 0$, the failure at $\vec{k} = 0$, a set of measure 0, does not screw up the wave packet.

Ground State Energy, Ground State Wavefunction

In most QM courses these are discussed in a model long before scattering. You usually use time independent perturbation theory. I'll show you how to get these quantities out of the time dependent perturbation theory we have already developed.

Why is the ground state energy interesting? We have been studying the response of the meson field to a classical source. In meson-"nucleon" theory, the source will be $\psi^*\psi$. Our classical source theory is a lot like a meson-"nucleon" theory with the "nucleons" nailed down. Take

$$\rho = \text{“}\delta''^{(3)}(\vec{x} - \vec{y}_1) + \text{“}\delta''^{(3)}(\vec{x} - \vec{y}_2)$$

The quotes are around the δ functions because we might want to smear them out a little bit. This is the charge density of two nucleons at $\vec{y}_1$ and $\vec{y}_2$. By computing the ground state energy and then by varying the positions we can find the potential between two "nucleons".

This is the same thing we do in QM. We calculate the interaction between the two protons due to their interaction with the electron in H_2^+ by considering how the ground state energy of the electron varies with the separation of the protons. The protons are nailed down in that calculation, usually you say that the protons are so much heavier than the electrons and move so slowly that we can treat the response of the electron field to changes in positions of the protons as if the changes take place adiabatically. Of course in that calculation we also have a Coulomb potential between the protons. Here we are trying to get at the whole internucleon potential by saying it all comes from the interaction with the meson field. Of course the Coulomb potential in QM really comes from the interaction with the photons...

Now to calculate $a = E_0$ by setting (3) = −(2):

$$(3) = -i \int dt f(t)(-a) = iaT\left(1 + O\left(\frac{\Delta}{T}\right)\right) = iE_0 T\left(1 + O\left(\frac{\Delta}{T}\right)\right)$$

$$(2) = \frac{(-ig)^2}{2!}\int d^4x_1 d^4x_f(t_1)f(t_2)\rho(\vec{x}_1)\rho(\vec{x}_2)\underbrace{\overbrace{\phi(x_1)\phi(x_2)}}_{\int \frac{d^4k}{(2\pi)^4}\frac{i}{k^2-\mu^2+i\epsilon}e^{ik\cdot(x_1-x_2)}}$$

$$= \frac{-ig^2}{2}\int \frac{d^3k}{(2\pi)^3}|\tilde{\rho}(\vec{k})|^2 \int \frac{dk^0}{2\pi}|\tilde{f}(\omega)|^2 \frac{1}{\omega^2 - \vec{k}^2 + i\epsilon}$$

Now $|\tilde{f}(k^0)|^2$ is sharply concentrated at $k^0 = 0$, we can replace ω in $\frac{1}{\omega^2-\vec{k}^2+i\epsilon}$ by 0 (and then the $i\epsilon$ is not needed any longer). Also

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi}|\tilde{f}(\omega)|^2 \underbrace{=}_{\substack{\text{famous theorem,}\\ \text{Parseval's theorem}}} \underbrace{\int_{-\infty}^{\infty} dt|f(t)|^2}_{T+O(\Delta)} = T\left(1+O\left(\frac{\Delta}{T}\right)\right)$$

We could sum up these properties by saying something sloppy like

$$\lim_{T\to\infty} |\tilde{f}(\omega)|^2 = 2\pi T\delta(\omega)$$

but what I have just shown is all (no more, no less) than that sloppy statement means.

$$(2) = \frac{ig^2}{2}T\left(1+O\left(\frac{\Delta}{T}\right)\right)\int \frac{d^3k}{(2\pi)^3}|\tilde{\rho}(\vec{k})|^2\frac{1}{|\vec{k}|^2+\mu^2}$$

The moment of truth: Set (3) = −(2), the T's and i's cancel.

$$E_0 \underset{T\to\infty}{=} \frac{-g^2}{2}\int \frac{d^3k}{(2\pi)^3}|\tilde{\rho}(\vec{k})|^2\frac{1}{\vec{k}^2+\mu^2}$$

The potential has come out in momentum space. To convert it to position space,

$$\text{Define}\quad V(\vec{x}) = -g^2\frac{d^3k}{(2\pi)^3}\frac{e^{i\vec{k}\cdot\vec{x}}}{\vec{k}^2+\mu^2}$$
$$\text{Then}\quad E_0 = \frac{1}{2}\int d^3x d^3y \rho(\vec{x})\rho(\vec{y})V(\vec{x}-\vec{y})$$

Then $\frac{1}{2}$ is the usual factor found even in electrostatics from overcounting the interaction when integrating over all space. For the two nucleon charge density, there will be four contributions. Two will be the interaction of the nucleons with each other (canlled by the $\frac{1}{2}$).

$$\rho(\vec{x}) = \text{“}\delta''^{(3)}(\vec{x}-\vec{y}_1) + \text{“}\delta''^{(3)}(\vec{x}-\vec{y}_2)$$
$$E_0 = \underbrace{\text{something independent of } \vec{y}_1,\ \vec{y}_2}_{\substack{\text{If “}\delta'' \to \delta\text{, this part} \to \infty\\ \text{Same problem as the self-energy}\\ \text{of a charged sphere in E.D.}}} + V(\vec{y}_1-\vec{y}_2)$$

The usual procedure for the integration of spherically symmetric Fourier trnasforms, followed by a contour integration gives

$$V(r) = \frac{-g^2}{4\pi r}e^{-\mu r} \qquad r = |\vec{y}_1-\vec{y}_2| \qquad \text{Yukawa potential}$$

Looks like the Coulomb potential for $r \ll \mu^{-1}$ the Compton wavelength of the meson, and falls off rapidly for $r \gg \mu^{-1}$.

The force is attractive (between like charges) (because the particle mediating it has even integer spin) and short[1] ranged because the mediating particle is massive. This potential has some of the essential features of the real nuclear force. Of course it doesn't include the effect of the whole family of mesons in the real world of multi-meson processes, but with $\mu = m_\pi$ it is a start.

The ground state wavefunction, is of course not a position space wavefunction (the expansion of $|\psi\rangle$ into $|x\rangle$'s), it is an expansion of $|0\rangle_P$ into the basis states $|\vec{k}_1, \ldots, \vec{k}_n\rangle$ of the noninteracting theory.

To get the ground state wave function of model 2 using time dependent perturbation theory, we'll use the results of model 1.

Consider

$$\rho(\vec{x}, t) = \begin{cases} \rho(\vec{x})e^{\epsilon t} & t < 0, \quad \epsilon \to 0^+ \\ 0 & t > 0 \end{cases}$$

That is we turn it on very slowly, arbitrarily slowly so that at $t = 0$ we finally have the full interaction of model 2, then we turn the interaction off abruptly.

Consider the S matrix in this theory

$$\langle \vec{k}_1, \ldots, \vec{k}_n | U_I(\infty, -\infty) | 0 \rangle = \langle \vec{k}_1, \ldots, \vec{k}_n | U_I(\infty, 0) U_I(0, -\infty) | 0 \rangle$$

Since the interaction is turned on arbitrarily slowly, $U_I(0, -\infty)$ should turn[2] the bare vacuum into $|0\rangle_P$. $U_I(\infty, 0)$, the evolution by the free Hamiltonian alone, which is 1 in the interaction picture, does nothing on the left so

$$\langle \vec{k}_1, \ldots, \vec{k}_n | U_I(\infty, -\infty) | 0 \rangle = \langle \vec{k}_1, \ldots, \vec{k}_n | 0 \rangle_P$$

which is what we are after.

Now we can apply the results of model 1 (Oct. 21, *p.11*)

$$\begin{aligned} \langle \vec{k}_1, \ldots, \vec{k}_n | 0 \rangle_P &= \langle \vec{k}_1, \ldots, \vec{k}_n | U_I(\infty, -\infty) | 0 \rangle \\ &= e^{-\frac{|\alpha|}{2}} e^{\frac{i\beta}{2}} f(\vec{k}_1) \cdots f(\vec{k}_n) \\ f(\vec{k}) &= \frac{1}{(2\pi)^{3/2}} \frac{1}{\sqrt{2\omega_{\vec{k}}}} (-ig) \tilde{\rho}(\vec{k}, \omega_{\vec{k}}) \qquad\qquad |\alpha| = \int d^3k |f(\vec{k})|^2 \\ \tilde{\rho}(k) &= \int d^4x e^{ik\cdot x} \rho(x) = \int d^3x e^{-i\vec{k}\cdot\vec{x}} \rho(\vec{x}) \int_{-\infty}^0 dt e^{ik^0 t} e^{\epsilon t} \\ &\underset{\epsilon>0}{=} \tilde{\rho}(\vec{k}) \frac{1}{ik^0 + \epsilon} \underset{\epsilon\to 0}{\longrightarrow} -\frac{i}{k^0} \tilde{\rho}(\vec{k}) \end{aligned}$$

The probability for having n mesons is

$$P_n = e^{-|\alpha|} \frac{|\alpha|^n}{n!}$$

[1] M_W is much larger and the weak force is thus much shorter ranged.

[2] screw the phase factor

What is P_n for a point charge at the origin? That is

$$\rho(\vec{x}) \to \delta(\vec{x}) \quad \text{(not ``δ'' smeared, the limit of a real point charge)}$$

Well,

$$\tilde{\rho}(\vec{k}) \to 1 \quad \text{and}$$

this is bad news; at high k we have a UV divergence in the integral for $|\alpha|$.

$$|\alpha| = \int d^3k |f(\vec{k})|^2 \underset{\text{high } k}{\sim} \int d^3k \frac{1}{\omega_{\vec{k}}^3} \underset{\text{high } k}{\sim} \int \frac{k^2 dk}{k^3}$$

The integral is log divergent. Since $\langle N \rangle = |\alpha|$ (Oct. 21, *p.12*) we see that not only the energy of the field becomes infinite in the limit of a point nucleon, but the ground state flees Fock space.

These infinities are scary but not harmful.

Physically observable quantities like the S matrix and the internucleon potential are hearteningly sensible.

Even if we don't go to the limit "δ" $\to \delta$, but instead take the limit of massless mesons $\mu \to 0$, $\langle N \rangle = |\alpha| \to \infty$, this time because of a small k divergence. What about that?

Answer: So what if there are an infinite number of mesons in the ground state. An experimentalist will tell you he can only measure the existence of a meson down to some low energy, not arbitrarily lo. If there are 1,000,000 mesons with a wavelength between $\frac{1}{2}$ light year and 1 light year, so what. It might be a problem if there were an infinite amount of energy at small $\vec{k}$, but there isn't.

$$\langle H \rangle = \int d^3k |f(\vec{k})|^2 \omega_{\vec{k}}$$

This is manifestly positive. Seems to contradict Yukawa pot. result.

And the extra factor of $\omega_{\vec{k}}$ moderates the IR divergence. The integral is finite even as $\mu \to 0$. Same with

$$\langle \vec{P} \rangle = \int d^3k |f(\vec{k})|^2 \vec{k}$$

So it seems we have been lucky. In this simple theory, the divergences have restricted themselves to unobservable quantities. Maybe the divergences will break this quarantine in more complicated theories. In fact there is a surprisingly wide class of theories in which the divergences don't break the quarantine. They are called "renormalizable" theories.

Next: Mass renormalization.

Model 3 and Mass Renormalization

The ground state energy, in perturbation theory, of this system is not necessarily zero. In this theory, that is not enough to make the 1-particle states of the Hammiltonian equal in energy to the 1-particle states of the full Hamiltonian. Indeed, the energy of a "static nucleon" depended on its interaction with the meson field. Not only did the vacuum energy of a state with one static nucleon depend on g, the coupling, it depended on how smeared the nucleon was.

The change in energy of a particle due to its interaction with another field is called "mass renormalization." The cure for this disease is also called "mass renormalization."

Mass renormalization goes all the way back to hydrodynamics.

Suppose I have a ping pong ball, with mass equal to $\frac{1}{20}$ of the water it displaces.

$$m_0 = \frac{1}{20}\rho V$$

Elementary hydrostatics tells you that there is an upward force on the ping pong ball equal to g times the mass of the water it displaces. There is also the downward force of gravity on the ping pong ball itself. The net force on the ball is thus $19m_0g$ upward. Putting this in Newton's equation we have

$$m_0 a = 19m_0 g \qquad\qquad a = 19g$$

The ball accelerates upwards at $19g$.

This is nonsense, as anyone who has ever held a ping pong ball underwater knows. The ping pong ball may accelerate up fast, but not at $19g$. The answer is not friction. You can see that the ping pong ball is not accelerating with $19g$ even when its velocity is low nad friction or viscosity is negligible.

The answer is that in order to move the ping pong ball, you have to move some fluid. In order to accelerate the ping pong ball, you have to accelerate some fluid. Stokes solved the fluid motion around a sphere.

If a ball that displaces volume V is moving with velocity $\vec{v}$ through the fluid, the fluid flow has a momentum, in the same direction as the ball, of $\frac{1}{2}\rho V\vec{v}$. The total momentum of the system, ball and fluid, is thus

$$\frac{1}{2}\rho V\vec{v} + m_0\vec{v} = 11m_0\vec{v}$$

which we set equal to the force after taking $\frac{d}{dt}$

$$\frac{dp}{dt} = 11m_0 a = 19m_0 g = F$$

$$a = \frac{19}{11}g$$

See the derivation in Landau and Lifshitz, Fluid Mechanics, leading up to the problem on p.36, for a more detailed understanding of the problem.

More motivation for mass renormalization:

Two limits of classical field theory:

Point particle limit: Mass renormalization occurs

Example: GR

Point particle of mass m creates a gravitational field which itself has energy density and creates further gravitational field.

Classical field limit: Can read dispersion relation for low amplitude plane waves off of the quadratic part of the Lagrangian.

Since the quantum field theory will probably exhibit all the behavior of the worst classical limit, we better be prepared for mass renormalization.

Another example of renormalizaed perturbation theory:

You could try, in the statistical-mechanical theory of critical phenomena to calculate the critical temperature, as well as other properties of the system in terms of the microphysical parameters. However, you may be able to do computations much more easily if you trade in one of the microphysical parameters for the critical temperature.

The classical theory of the electron also suffers mass renormalization. Imagine the electron as a charged shell. The bare mass is m_0, the charge, e, the radius, r. There is a contribution to the measured mass of the electron other than m_0. There is the electrostatic energy (divided by c^2).

$$\underbrace{m}_{\text{measured, physical mass}} = m_0 + \frac{e^2}{2rc^2}$$

Model 3 is going to suffer mass renormalization. The energy of a single meson state or a single nucleon state is going to depend on its interaction. We looked at static "nucleons" interacting iwth the meson field in model 2. Recall that even for a single nucleon weighted down at $\vec{y}$

$$\rho(\vec{x}) = \text{“}\delta\text{”}^{(3)}(\vec{x} - \vec{y})$$

the energy of the system depends in detail on how we smear out the δ function. In fact if we don't smear it out at all

$$\rho(\vec{x}) = \delta^{(3)}(\vec{x}) - \vec{y})$$

the energy of the meson ground state $\to -\infty$. Now the energy of a one nucleon state includes the change in energy of the meson field its presence causes, and although the features of this effect mass change when the coupling to the ϕ field goes from $\rho(\vec{x})\phi(x)$ (nailed down nucleons) to $\psi^*\psi\phi(x)$ (dynamical nucleons) there is no reason to expect it to go away.

This is going to be bad news for scattering theory. Just as the failure to match up the ground state energy for the noninteracting and full Hamiltonians in model 2 produced T and Δ dependent phases in $\langle 0|S|0\rangle$, the failure to match up one particle state energies in model 3 will yield T and Δ dependent phases in $\vec{k}|S|\vec{k}'\rangle$. Worse than that, it can even cause two wave packets that were arranged to collide, not to collide. I'll show how:

In our relativistic interacting theory, for sufficiently weak coupling, we expect that there will be one nucleon states, $|\vec{p}\rangle_P$, which are eigenstates of the full Hamiltonian, H:

$$H|\vec{p}\rangle_P = \sqrt{\vec{p}^2 + m^2}|\vec{p}\rangle_P$$

There are also eigenstates of the Hamiltonian, which because of this mass renormalization mess, can have a different mass, If I prepare a packet of these free Hamiltonian eigenstates they propagate along with group velocity

$$\vec{v} = \frac{\partial E}{\partial \vec{p}} \left(= \frac{\vec{p}}{E} \quad \text{when} \quad E^2 = \vec{p}^2 + m^2 \right)$$

When I turn the interaction on (slowly, so that the free Hamiltonian eigenstate (bare nucleon) turns into the dressed nucleon (full Hamiltonian eigenstate)) the group velocity changes because the mass in $E = \sqrt{\vec{p}^2 + m^2}$ changes. I could set up a nucleon and meson to scatter, and if I turn on the interaction too early or too late, they might not even come close!

To fix up this problem, we are going to introduce new counterterms in our theory

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - \frac{\mu^2}{2}\phi^2 + \partial_\mu\psi^*\partial^\mu\psi - m^2\psi^*\psi + f(t)\Big[-g\psi^*\psi\phi + \underbrace{a}_{\substack{\text{vacuum energy}\\ \text{density counterterm}}} + \underbrace{\frac{b}{2}\phi^2}_{\text{meson mass counterterm}} + \underbrace{c\psi^*\psi}_{\substack{\text{"nucleon"}\\ \text{mass counterterm}}}\Big]$$

μ is the measured mass of the meson. m is the measured mass of the nucleon.

When the interaction is off ($f(t) = 0$) this theory is a free theory with mesons of mass μ and nucleons of mass m.

When the interaction is turned on ($f(t) = 1$), we arrange, by adjusting b and c, that the one meson state has mass μ and the one nucleon state has mass m. This eliminates the phases in the one particle matrix elements by matching the energy of the one particle state (the vacuum energy, which is an infinite volume system may be infinite, being proportional to the volume, is adjusted to zero with the help of a).

To summarize, the conditions determining a, b, and c are

$$\langle 0|S|0\rangle = 1 \Rightarrow a$$

$$\langle \overbrace{\vec{q}}^{\text{one meson states}} |S| \overbrace{\vec{q}\,'} \rangle = \delta^{(3)}(\vec{q} - \vec{q}\,') \Rightarrow b$$

The one meson and one nucleon states shouldn't do anything; they have got nothing to scatter (only vacuum)

$$\langle \overbrace{\vec{p}}^{\text{one nucleon states}} |S| \overbrace{\vec{p}\,'} \rangle = \delta^{(3)}(\vec{p} - \vec{p}\,') \Rightarrow c$$

This procedure should match up the energies of states of widely separated nucleons and mesons, without any additional twiddling. The energy of two widely separated mesons, even in model 3 when they are affected by self-interaction, should be the sum of their respective energies. Matching the energies of the vacuum and the one particle state matches the energy of states of widely separated mesons too. Although the particles in model 3 never become separated from their own fields, in the far past / future, they always become widely separated from each other.

Sometimes it is useful to think about

$$\mu_0^2 \equiv \mu^2 - b \quad \text{and} \quad m_0^2 \equiv m^2 - c$$

the coefficients of $\frac{1}{2}\phi^2$ and $\psi^*\psi$ in the full Lagrangian, respectively, although they have very little physical significance. What our procedure amounts to is breaking up the free and interacting parts of the Hamiltonian in a less naive way. You are always free to break up the free and interacting parts of the Hamiltonian any way you like, although you won't get anywhere unless you can solve the free Hamiltonian.

We have put $\frac{b}{2}\phi^2$ and $c\psi^*\psi$ in with the interaction $g\psi^*\psi\phi$ because that way the mass of the meson (nucleon) is $\mu(m)$ when the interaction is of (manifestly) and the mass of the meson (nucleon) is $\mu(m)$ when the interaction is on (by our careful checks of b and c).

This procedure gives us a BONUS.

By making b and c (hence μ_0^2 and m_0^2) quantities you compute, our perturbation theory is expressed in terms of the actual physical masses, not the dumb quantities, μ_0 and m_0.

If you treated m_0 and μ_0 as fundamental, you would calculate all your cross sections, bound state energy levels, all quantities of interest, in terms of them, and them to make contact with reality, you would have to calculate μ and m, the physical masses, in terms of m_0 and μ_0 too. Since no one is interested in your m_0 and μ_0, to present your results, you would have to reexpress all your cross sections in terms of μ and m.

We have bypassed that mess by turning perturbation theory on its head. Instead of a perturbation theory for m^2, μ^2 and all other physical quantities in temrms of m_0^2 and μ_0^2, we have a perturbation theory (for m_0^2 and μ_0^2) and all physical quantities in terms of the observed masses, m and μ.