Exam – due January 12th, 11:59:59 pm: 60 hours

Use any notes [except for papers or sections of books where the exact same problems are solved if you find such sources] or books but don't collaborate. Ask questions at baumgart or motl at fas.harvard.edu or 617-384-9488 (office) or 617-868-4487 (home). Corrections if any will be distributed by e-mail. Submit the solutions in L.M.'s Lyman mailbox [by Friday evening/midnight]. If you can't get to the Jefferson building before the midnight etc., you may also try 1 Chauncy St. apt 18 (ring the bell).

Exam 1: A simple quartic interaction

Consider two real scalar fields with somewhat unusual kinetic term that also includes a mixed term:

$$\mathcal{L} = \frac{1}{2}(1+K^2)\partial_\mu\phi_1\partial^\mu\phi_1 + \frac{1}{2}\partial_\mu\phi_2\partial^\mu\phi_2 + K\partial_\mu\phi_1\partial^\mu\phi_2 - \lambda(\phi_1)^2(\phi_2)^2$$

where K, λ are positive real numbers and the last term is an interaction term. Solve the following problems:

- Write the kinetic terms – derivative bilinear terms – for the scalar fields using a 2×2 matrix for both of them.
- Recall that the propagators are given as the inverse of the bilinear terms in the Lagrangian. What is $\langle 0|T[\phi_i(x)\phi_j(y)]|0\rangle$ for $i,j=1,2$ as a Fourier transform?
- Introduce better fields χ_1,χ_2 as functions of ϕ_1,ϕ_2 that have the ordinary kinetic terms, and write down the full Lagrangian in terms of χ_1,χ_2. Choose $\chi_1=\phi_1$ if you can.
- At the lowest order in λ, compute the matrix element

  $$\langle \vec{p}_1, \vec{p}_2|S|\vec{k}_1, \vec{k}_2\rangle$$

  where the indices 1, 2 of the momenta tell you whether the particles are created by χ_1 or χ_2. Write down what normalization of the two-particle states you used i.e. what is

  $$\langle \vec{p}_1, \vec{p}_2|1|\vec{k}_1, \vec{k}_2\rangle.$$

- At the lowest non-trivial order in λ, calculate

  $$\langle \vec{p}_2, \vec{q}_2|S|\vec{j}_2, \vec{k}_2\rangle$$

  Note that all particles are those created by χ_2 now. You may need loops. Introduce a convenient renormalization scheme for λ and write the finite scattering amplitude in this scheme.

Exam 2: Interactions of fermions

Consider a simple theory of a massive Dirac fermion with a quartic interaction:

$$\mathcal{L} = \bar\psi(i\partial\!\!\!/ - m)\psi - \frac{1}{2}\bar\psi(a+b\gamma_5)\psi\cdot\bar\psi(c+d\gamma_5)\psi$$

where a,c are real constants while b,d are pure imaginary. Work in the center-of-mass frame.

- Is the interaction renormalizable according to its dimension?

- Calculate the total $e^+e^- \to e^+e^-$ cross section at the tree level as a function of θ, the scattering angle. In doing the amplitudes, be careful about the relative signs. Take the average over the initial spin polarizations and sum over the final polarizations.

- Calculate the difference between the total cross section to $0 < \theta < \pi/2$ minus the total cross section to $\pi/2 < \theta < \pi$. What do the values of a, b, c, d have to satisfy for you to know that the difference has to vanish (to all orders), using a symmetry argument? You may also decide that the previous question relies on a wrong assumption: in this case, explain why. Try to impose the weakest possible conditions and/or list all possibilities when it's zero. Check that it vanishes if these assumptions are satisfied.

Exam 3: Box diagram in QED

In 253b, you will learn about a new kind of fields – the spin 1 vector fields such as the electromagnetic field. They will also interact with the fermions. Instead of the Yukawa interaction, you will consider the gauge interaction

$$\mathcal{L}_{int} = -e\bar{\psi}\gamma^\mu\psi A_\mu.$$

But the machinery won't be too different. The electromagnetic field A_μ and its generalizations will have propagators that will carry a Lorentz vector index such as μ at both endpoints. They can be taken to be proportional to $g^{\mu\nu}$ but you won't need them here anyway. The vertex from the interaction above will correspond simply to

$$-ie\gamma^\mu.$$

The rules how you follow the fermion propagators against the arrows are not changed. In this theory of a Dirac field ψ whose mass is m and the electromagnetic field A_μ, called Quantum Electrodynamics (QED), consider the one-loop correction to the four-point function of four fields $A_{\mu_i}(x_i)$, $i = 1, 2, 3, 4$, in which an electron is running in the loop. So the diagram is a square – a box – with four attached photon lines. You can really imagine that the indices μ behave just like internal indices labeling four different scalar fields A_0, A_1, A_2, A_3 except that γ^μ appears in the vertex.

- By using the naive superficial degree of divergence, what kind of divergence or convergence do you expect from this diagram?

- What is the relation between the complete answer to the previous question – "the sum of the diagrams is convergent/divergent" – and the question whether QED is renormalizable? Note that QED only has the free (bilinear) terms for the electron and photon fields and the cubic interaction.

- Now, the superficial degree of the divergence may be too pessimistic. Start to calculate whether the divergence is really there if you answered "divergent" in the first task, and try to find a neglected divergence of a subgraph if you answered "convergent". Write down the momentum integral defining the correction to the photon four-point function (without the external propagators!) and include all possible permutations (orderings that can't be obtained from each other by cyclical permutations of the photon lines attached to the fermion loop) that contribute at the same order. You will need the fermion propagators in the form $(\not{k} + m)/(k^2 - m^2 + i\epsilon)$.

- Take the numerator of the integrand. Focus on the leading terms in k. For example, if you found that there is a potential divergence, these terms are those that can lead to the divergence. The leading terms in the numerator should be a trace of the product of eight factors, each of them being either γ_{μ_i} or $\not{k}$. Include all the permutations you need. Don't waste too much of your time with the overall normalization but be careful about the relative normalizations.

- Compute these traces, using the tricks like $\gamma_\mu \not k = -\not k \gamma_\mu + 2k_\mu$ and the known traces of products of gamma matrices etc. A long calculation should follow here... What you're really interested is the coefficients of the leading term as a function of (large) k_ν. For example, if you found out that there are candidate divergences, you're interested in the coefficient of the leading divergence. What is the sum? What does it mean for the renormalizability of QED?

  Hint: If you dislike abstract formulae with a lot of products of $g^{\mu\nu}$, consider the integrand, without a loss of generality, at $k^\mu = (0, 0, 0, k)$ (in the direction 3, possible by a rotation in the Euclidean momentum space). Explain why the indices $\mu_1 \dots \mu_4$ must come in pairs for the result to be nonzero and why $\mu_1 \dots \mu_4 = 1111, 1122, 1133, 3333$ and their permutations are the only really inequivalent possibilities. In each of these four cases, you want to obtain a certain result which you don't at the beginning, only after you integrate over d^4k which can be done if you now reconstruct the relative coefficients of the terms like $g_{\mu_1\mu_2} g_{\mu_3\mu_4}$.

- Regardless of the previous result, assume that the sum over the different cyclical orderings is actually finite i.e. the sum is convergent. By dimensional analysis, what is the correction to the four-point function up to a numerical coefficient? Write down the correct power law dependence on e and the mass m. Assume that the angles between the photons are of order one and all of them have energies of order E but assume that there is no dependence on E.

- What is the typical energy scale of the energy of the photons for which you expect the (unusual) photon-photon scattering to be strong, i.e. what is the value of E, up to a numerical constant, for which the previous estimate is correct? What is your estimate for the total cross section of the light-light scattering (four external photons), up to a numerical constant? Again, you only want the right powers of e and m and dimensional analysis is OK. Compute the result in femtobarns (10^{-43} squared meters) assuming the typical energies at which the process is maximally frequent. Everywhere, work in supernatural units where $1 = 2 = \pi$ etc.

Warning: the numerical constants such as powers of π etc. can be quite significant here and the dimensional analysis estimate may be quantitatively misleading.

Exam 4: Sum of integers: regularization

Consider a real scalar field in a $1 + 1$-dimensional spacetime. Let the spatial dimension be periodic so that the momentum is quantized. For every integer momentum n, there will be a creation and annihilation operator $a_n, a_n^\dagger$ that satisfy

$$[a_m, a_n^\dagger] = \delta_{mn}, \qquad m, n = 1, 2, 3, \dots$$

with other commutators equal to zero. We will ignore the mode with $m = 0$ as well as those with negative m (because they are analogous). We will normalize the Hamiltonian as follows:

$$H = \sum_{n=1}^{\infty} \frac{n}{2} \left(a_n^\dagger a_n + a_n a_n^\dagger \right).$$

You see that we obtained the energy of a quantum proportional to n and the two different orderings appear with the same coefficient.

- Write down the Hamiltonian as a sum of a normal-ordered expression plus the ground state energy. Is the ground state energy convergent? You will have to find its "finite value".

- Add a factor of $\exp(-\lambda n)$ to each term of the sum. What is the limit for λ in which you reproduce the result without this exponential factor? Compute the exact sum for any λ and expand it into a Laurent expansion around the relevant value of λ, keeping two leading non-zero terms.

The divergent part can be removed by a counterterm but the finite part has an absolute meaning and should be taken seriously. Compute the finite part in one more independent way. The method below is analogous to the dimensional regularization while the method above resembles a momentum cutoff.

- Write the ground-state energy as a multiple of

$$\zeta(x) = \sum_{n=1}^{\infty} \frac{1}{n^x}$$

  for an appropriate value of x. What is the value of x that we're interested in?

- Define a more general function

$$\zeta_s(x) = \sum_{n=1}^{\infty} \frac{1}{(n+s)^x}.$$

  For what value of s does it reduce to $\zeta(x)$? For what values of x is the sum convergent?

- Find a simple additive relation between $\zeta_s(x)$ for $s = 0$ and $s = 1$, assuming that the sum converges. Also, write $\zeta_1(x)$ as the Taylor expansion of $\zeta_s(x)$ around $s = 0$ with respect to s. Evaluate the m-th derivative of the generalized zeta function with respect to s at $s = 0$ in terms of $\zeta_s(x)$. Combine all results here to find a relation between $\zeta_0(x)$ for different values of x. For certain integer values of x, only a finite number of the terms in the Taylor expansion will be nonzero.

- Substitute an appropriate value x_0 of x into this relation to see that $\zeta_0(1+y) = 1/y + O(1)$. Then substitute value $x_0 - 1$ for x to calculate $\zeta_0(0)$, and finally substitute $x_0 - 2$ to calculate $\zeta_0(-1)$. Be deliberately sloppy about the fact whether the expressions are convergent. Compare your new result for the sum with the previous result.