Homework 4 - due Friday, Oct 20th

Read Coleman’s notes and Chapter 4 of Peskin & Schroeder.

(4A) Nonrelativistic complex scalar: second quantization

Imagine that the non-relativistic Schrodinger equation
\[ i\frac{\partial}{\partial t} \psi = -b \nabla^2 \psi \]
is an equation of motion for a scalar field \( \psi(x, t) \). Here, \( b = 1/2m \) in units of \( \hbar = 1 \).

- Write down the Lagrangian and action that gives the correct field equation for \( \psi \).
- Vary the action with respect to \( \psi \) and derive the equation of motion for \( \psi^* \). Do they make sense? Would the equations change if you took the real part of the action only?
- Construct the stress-energy tensor by the Noether procedure from the translation symmetry; you will have to treat the time component and spatial components separately. Write down the integrals for the total momentum and the total energy, \( H \) and \( \vec{P} \).
- Construct the current for the symmetry \( \psi(x) \rightarrow \psi(x)e^{i\alpha} \) for a constant \( \alpha \), and the corresponding charge \( Q \).
- Expand \( \psi \) and \( \psi^* \) into the most general plane waves that solve the equations of motion. Call the coefficients of the plane waves \( \alpha_k \) for \( \psi \) and \( \alpha_k^\dagger \) for \( \psi^* \). Be careful about the relation between the energy and the momentum.
- Determine the canonical momentum for \( \psi \) from the action, and write down the commutation relations between \( \psi(x) \) and \( \psi^*(y) \) as well as pairs of \( \psi \) and pairs of \( \psi^* \). Do you need to know \( \partial \psi/\partial t \) to fully specify the initial conditions?
- Write down \( H, \vec{P}, Q \) in terms of \( \alpha \) and \( \alpha^\dagger \) and try to interpret it or justify that it makes sense.

(4B) The model one with the linear interaction

Take the model 1 from Coleman’s page 103
\[ \mathcal{L} = \frac{1}{2} \partial_\mu \phi \partial^\mu \phi - \frac{\mu^2}{2} \phi^2 - g \rho(x) \phi(x) \]
where \( \rho(x) \) depends on four coordinates \( x^\mu \) and is treated as a set of coupling constants for an interaction – parameters of the theory. Assume that it decreases sufficiently quickly at infinity, both in space and in time. Recall what Dyson’s formula is, and all other material you will need. Also, define \( \check{\rho}(k) \) as the 4D Fourier transform of \( \rho(x) \).

- Assume that the initial state is the vacuum state with no particles. Calculate the amplitude that it will end up as a vacuum up to the zeroth order in the constants \( \rho(x) \).
- That was too easy. What is the lowest order in \( \rho(x) \) at which the amplitude of going from the vacuum to the vacuum will be corrected?
• That was still too easy. Now actually calculate the vacuum-vacuum amplitude up to this order in $\rho(x)$ and don’t forget the leading contribution: introduce the symbol $-\lambda/2$ for the first correction to the vacuum-vacuum amplitude.

• What is the probability of the vacuum evolving into a vacuum up to this order?

• Go to higher orders, and still calculate the vacuum-vacuum amplitude. Which orders in $\rho$ will contribute? Be careful about the combinatorial factors from the Wick’s theorem and/or from the Feynman rules that you introduced.

• Take the full vacuum-vacuum amplitude you have found now and calculate the vacuum-vacuum probability. That was easy.

• Now try to calculate the amplitude that the initial vacuum state evolves into a one-particle state with momentum $k$ to all orders in $\lambda$, and compute the total probability to create 1 particle with any momentum via an integral. Note that the Feynman diagrams will be products of something new and something that you have already calculated.

• Calculate the amplitude for the vacuum to evolve into $m$ particles with some momenta, and the corresponding probability with integrated momenta.

• Check that the total probability is one, and try to fix the errors in your previous calculation to make it true. Hint: Poisson distribution.