(6A) Decay

Write down the simplest Lagrangian that admits the following process: a real scalar particle $\phi$ of mass $m$ directly decays into three real massless particles – excitations of the field $\chi$. Normalize the interaction term to make the Feynman rules as simple as possible, and use a factor of $-g$ in this interaction term of the Lagrangian (times an appropriate rational factor to make it simple). Check that $g$ is dimensionless.

Denote the energies $p^0$ of the final three particles to be $E_B, E_C, E_D$ and their 3-momenta $\vec{p}_B, \vec{p}_C, \vec{p}_D$. In the two-dimensional plane with coordinates $E_B, E_C$, what is the kinematically allowed region of $E_B, E_C$ in the decay? Finally, compute the total decay rate $\Gamma$.

(6B) Feynman rules for a derivative theory

Consider the Lagrangian
\[ \mathcal{L} = \frac{1}{2} \partial\mu \phi \partial\mu \phi - \frac{u}{4!} \partial\alpha \phi \partial\beta \phi \partial\gamma \phi. \]
Here, $u$ is a coupling constant. What is the mass dimension of $u$ in units $c = \hbar = 1$? What is the mass of the field $\phi$? Write down the Feynman rules for this theory, especially for the vertex proportional to $u$. What counterterms do you need up to the order $u^2$ to preserve the vanishing energy of the vacuum and the previously determined mass of $\phi$? Don’t calculate their values.

(6C) A simple phase space integral

In $d$ spacetime dimensions, the Lorentz-invariance phase space for one particle is proportional to the factor
\[ d\Omega = \frac{d^{d-1}k}{(2\pi)^{d-1}E_k} \]
where $E_k = (\vec{k}^2 + m^2)^{1/2}$ where $m$ is the mass. Calculate the total volume $\Omega$ of the region of this phase space bounded by $E_k < E$ where $E$ is a constant upper cutoff for the energy. What is the power law between $\Omega$ and $E$ for large values of $E$?

(6D) Generalizing Mandelstam variables

Consider $N$ different spacetime momenta $p_i^\mu$ where $i = 1 \ldots N$ that satisfy
\[ \sum_{i=1}^{N} p_i^\mu = 0 \]
Each $p_i^\mu$ is on-shell with a different mass, $p_i^\mu p_i,\mu = m_i^2$ for $i = 1 \ldots N$. Define variables
\[ s_{ij} = (p_i + p_j)^2 \]
where the Minkowski $++--$ inner product is included in the right-hand side. How many different variables $s_{ij}$ for $i \neq j$ there are? Don’t double count. (For $N = 4$, you get the Mandelstam variables back.) Calculate the sum of all nontrivial values $s_{ij}$ in terms of the masses $m_i$. Hint: if the sum of the momenta is zero, its square is still zero.