Homework 7 – due Friday, Nov 10th – 7A,7B,7C

(7A) Quartic interaction

Consider a complex scalar field with a quartic coupling:

$$\mathcal{L} = \partial_\mu \psi^* \partial^\mu \psi - m^2 \psi^* \psi - \lambda \frac{1}{4} (\psi^* \psi)^2$$

Think about the Feynman rules; ignore counterterms. What is the invariant amplitude for an $N + \bar{N} \to N + \bar{N}$ elastic scattering at order $\lambda$? The initial four-momenta are $p_1, p_2$, the final momenta are $p'_1, p'_2$.

Write down the integral for the contribution to the same amplitude at order $\lambda^2$ but don’t fully evaluate it. However, rewrite the integral in terms of one Feynman parameter and one Euclidean four-momentum.

Find the imaginary part of this amplitude and show that it is consistent with the optical theorem.

(7B) Symmetry breaking

Consider the Lagrangian with $N$ complex scalar fields

$$\mathcal{L} = \sum_{a=1}^{N} (\partial_\mu \psi^{a*} \partial^\mu \psi^a - m^2 \psi^{a*} \psi^a) - \frac{1}{4} \lambda \left( \sum_{a=1}^{N} \psi^{a*} \psi^a \right)^2$$

Does the theory have a $U(N)$ symmetry? For fixed indices $a, b, c, d$, compute the invariant amplitude for the scattering process

$$\psi^a + \psi^{b*} \to \psi^c + \psi^{d*}$$

where the letters represent the particles created by the fields denoted by the same letter. You will need a lot of Kronecker deltas.

Now assume that $m^2$ is negative. That would look like a theory that is inconsistent or contains superluminal tachyons. But actually the theory will look fine if you replace the last field $\psi^N$ by two new real Klein-Gordon fields $h, \pi$ by

$$\psi^N = v + \frac{h + i\pi}{\sqrt{2}}$$

Here, $v$ is a constant. Rewrite the Lagrangian in terms of $h, \rho, \psi^1...^N-1$, and the $\psi^*$ conjugates. For what value of $v$ (the “vacuum expectation value”) do you cancel the linear terms in $h$ and/or $\pi$? What is the manifest unbroken symmetry of your new theory – i.e. the symmetry preserving both the Lagrangian and the configuration where $\psi^a = (0, 0, 0, \ldots, v)$? Compute the (real) dimensions of the original symmetry group and the new one, and their difference $D'$. Also, count the number of (real) massless fields $G'$ (one complex is two real) in the theory involving $h, \pi$. Compare $D'$ and $G'$.

What is the mass of $h$?

The fact that this configuration (new vacuum) preserves a smaller symmetry than the Lagrangian is called a spontaneous symmetry breaking: the vacuum has spontaneously “rolled down” into a smaller value of the potential energy where some of the symmetry of the Lagrangian is broken: it’s just like the egg standing on its tip. It will roll and the $SO(2)$ symmetry will be broken at the end.

Now assume that the value of $v$ is what you have found, and calculate the scattering amplitude

$$h + \psi^e \to h + \psi^f, \quad e, f = 1 \ldots N - 1$$

at the tree level. If you wish, you can think about the complete Feynman rules. In what regime of energies of the particles would you expect the $he-hf$ amplitude reproduce the $ab-cd$ amplitude at the beginning? Can you check this statement?
(7C) Five-point function

Consider a real scalar field with a cubic interaction:

\[ \mathcal{L} = \frac{1}{2} (\partial_\mu \phi)^2 - \frac{\mu^2}{2} \phi^2 - \frac{\lambda}{3!} \phi^3 \]

Compute the connected five-point correlation function \( G(p_1, p_2, p_3, p_4, p_5) \) in the tree-level approximation. How many non-trivial shapes of the Feynman diagrams (with all labels removed) are there? How many different copies with different dependences on the five momenta do you have to sum?

Add the term \( J(x) \phi(x) \) in the Lagrangian density. If we define the generating functional as in class

\[ Z[J] = \langle 0 | \mathcal{S} | 0 \rangle_p, \]

how do you express \( G(p_1 \ldots p_5) \) in terms of derivatives acting on \( Z[J] \)? Be careful about logarithms, exponentials, and connectedness.