(10A) Hydrogen atom and $SO(4)$ symmetry

**Task:** Find the energy spectrum of all bound states of the Hydrogen atom in non-relativistic quantum mechanics and their degeneracy purely by symmetry considerations. The Hamiltonian is

$$H = \frac{\mathbf{p}^2}{2m} - \frac{k}{r}, \quad k \text{ is a constant}$$

**Hints:**

- Define the orbital angular momentum

$$L_i = \epsilon_{ijk}x_jp_k, \quad \epsilon_{123} = +1$$

and compute the commutators $[L_i, L_j]$ i.e. the Lie algebra. What is the commutator of $L_i$ with $H$?

- Define the Runge-Lenz vector

$$\vec{A} = \frac{1}{mk}\vec{L} \times \vec{p} + \frac{\vec{x}}{r}.$$ 

Is there any ordering ambiguity in the cross-product above? Compute the commutators $[L_i, A_j], [A_i, A_j], [A_i, H]$.

The last commutator tells you, in the classical limit, whether the elliptic orbit rotates or not. Write down the $[A_i, A_j]$ commutator as $H$ times something. (Anti)symmetrization is a good bookkeeping device for many terms.

- Now prove that the six generators that are components of $\vec{L}, \vec{A}$ generate a Lie algebra isomorphic to $SO(4)$. In other words, find the relation between $J_{IJ}, I = 1, 2, 3, 4$, on one side, and components of $\vec{L}, \vec{A}$ on the other side, such that the commutators translate into each other. Note that $J_{IJ}$ is a matrix with $+i$ on the position $(I, J)$ and $-i$ on the position $(J, I)$ and otherwise filled with zeroes.

Work in a formalism in which a $SO(3)$ subgroup of $SO(4)$ is manifest.

- Now show that this $SO(4)$ Lie algebra is isomorphic to $SU(2) \times SU(2)$, two mutually commuting copies of the usual $SU(2)$ algebra. Warning: the previous $SO(3)$ subgroup is neither of the $SU(2)$’s here. This isomorphism is important for many considerations, for example in the Euclidean four-dimensional spacetime (with imaginary time). Hint: define the generators of $SU(2)_M \times SU(2)_N$ as

$$M_i = \alpha L_i + \beta A_i, \quad N_i = \alpha L_i - \beta A_i$$

in such a way that $[M_i, N_j] = 0$ and $[M_i, M_j]$ as well as $[N_i, N_j]$ behave like the normal $SU(2)$ algebras. The “numbers” $\alpha, \beta$ will be determined and they can depend on the energy $E$: once you study a particular energy level, there can still be a representation of the $SO(4)$ symmetry living at this level but the operator $H$ is equal to a number at this subspace of the Hilbert space.

- Prove that $\vec{M}^2 = \vec{N}^2$. What does it mean for $\vec{L}, \vec{A}$? Write the eigenvalue of $\vec{M}^2 = \vec{N}^2$ in terms of $j = 0, 1/2, 1, 3/2, \ldots$. What is the degeneracy of the multiplet as a function of $j$, and as a function of $n = 2j + 1$?

- Compute $H(M^2 + \gamma)$ where $\gamma$ is a constant (like $1/4$) such that the result is simple. Using this result, write $H$ (the energy eigenvalue) as a function of $M^2$ and using the previous result, write down the allowed values of $n = 2j + 1$, the allowed negative energy eigenvalues, and the degeneracies of the corresponding levels.