Homework 11 (files hwb*.*) – due Tuesday, Dec 12th

(11A) Dirac spinors in $d$ dimensions

Consider $d$-dimensional spacetime with signature $(d-1,1)$. Start with $d$ even, $d=2n$. Find an appropriate minimum set of matrices that are expressed as tensor products of Pauli matrices or the identity matrix that satisfy the correct algebra

$$\{\gamma^\mu, \gamma^\nu\} = 2\eta^\mu\nu = 2 \text{ diag } (+ - - \ldots - -)$$

Hint: you may use the following type of the tensor products

$$1 \otimes 1 \otimes \ldots 1 \otimes \sigma_{x,y} \otimes \sigma_z \otimes \ldots \otimes \sigma_z \otimes \sigma_z$$

where $\sigma_{x,y}$ is either $\sigma_x$ or $\sigma_y$: your ensemble of the matrices will have one-half with $x$ and one-half with $y$. As the matrices $\sigma_z$ are added on the right, the matrices 1 are eaten from the left. You will also need to add a factor of $i$ for some of the matrices. Recall that the algebra means that each pair of gamma matrices anticommutes, and the square of each should be plus minus the identity matrix.

Fermionic harmonic oscillator representation

Also, write down a different representation of the gamma matrix anticommutator in terms of $(a_i + a_i^\dagger)$ and $i(a_i - a_i^\dagger)$ where $a_i, a_i^\dagger$ are annihilation and creation operators on a fermionic Harmonic oscillator (where the commutators are replaced by anticommutators). How many dimensional harmonic oscillator do you need? This is not a task but try to see why this representation is equivalent to the tensor products of Pauli matrices.

Dimensions of representations

Back to the tensor products. What is the size of these matrices i.e. the dimension of the Dirac spinor representation for $d=2n$? Write the result for a few values of $n$. Which of the matrices are Hermitean?

Chirality and Weyl spinors

Now, for $d=2n$, define the matrix of chirality

$$\tilde{\gamma} = e^{i\alpha} \gamma^0 \gamma^1 \ldots \gamma^{2n}$$

with such a phase that

$$\tilde{\gamma}^2 = 1.$$ 

What are the eigenvalues of $\tilde{\gamma}$ and how many dimensions the corresponding eigenspaces have? You will need to count transpositions to see what phase you need. Does $\tilde{\gamma}$ commute or anticommute with the $\gamma^\mu$ matrices? Show that $\tilde{\gamma}$ commutes with the Lorentz generators

$$J^\mu\nu = \frac{i}{4}(\gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu).$$

This fact can be used to decompose the Dirac representation to two irreducible – Weyl (=chiral spinor) – representations according to the eigenvalues of $\tilde{\gamma}$.

The space of all operators on the spinor space

How many complex dimensions are there in the space of all possible matrices acting on the Dirac spinors? Write down a natural basis of this space in terms of the gamma matrices and their products, and check the counting of the total number of the matrices.
Spinors in odd dimensions

Now, use your results for $d = 2n$ dimensions and find matrices that satisfy

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} = 2 \text{diag} (+ - - \ldots - -)$$

in $d = 2n + 1$ dimensions without any extra work. You will need one more gamma matrix with the right anticommutation rules with the old gang but you will be able to define it in terms of one of the matrices found above. Be careful about the phase although it is not unique. Message: the Dirac spinor in $2n$ dimensions can also be used as a spinor in $2n + 1$ dimensions. But in $2n + 1$ (odd) dimensionalities, there is no notion of chirality: that’s related to the fact that the disconnected component of $O(2n + 1)$ and its non-compact versions can be written as $-1$ times the elements of the elements of $SO(2n + 1)$ or its versions which means that you don’t need to add another “Weyl” spinor in order to find a representation for the parity symmetry as you needed in the even total dimensions.

**Dirac conjugate and reality conditions**

Construct the Dirac conjugate of a spinor, $\bar{\psi}$, as $\psi^\dagger$ with an extra matrix so that the products $\bar{\psi}_1 \psi_2$ are Lorentz-invariant. This is not hard. Finally, as a bonus for the brave students, try to find out whether the Weyl (smaller, chiral) representations of the Lorentz group are real, complex, or quaternionic, as a function of the dimension. The answer is a periodic function of the dimension with period eight.