Astro 503 Homework #10

Due Thursday, April 6

1. **The Ephemeris:** You now each have your code assignments and a good idea of the `main()` requirements. To finish this up: submit by this Thursday (April 6) the code for your part of the assignment. Also, between the group of you, write a driver program for each of the 3 use cases, and put those in my hands by April 6.

   I will put all of the code in a group-accessible place, so that you can each compile and link the programs. Then you will have until the following Thursday **(April 13)** to get the programs working properly. If you find a flaw in the behavior of someone else's code, you'll have to notify them and it'll be their job to fix it.

2. **Dark Energy:** Here you will calculate the size and age of the Universe as a function of cosmological parameters. You will need to do some numerical integrals; you are, as usual, free to make use of Numerical Recipes routines, though I strongly suggest that you look at each one and update it to reflect good C++ practice.

   Most of you know this, but: the Robertson-Walker metric for a homogeneous isotropic Universe is

$$ds^2 = c^2 dt^2 - a^2(t)\left[d\chi^2 + D^2(\chi)(d\theta^2 + \sin^2\theta d\phi^2)\right]. \tag{1}$$

   Here χ is the *comoving radial coordinate*, $a(t)$ is called the *scale factor*, and the function D depends upon the radius of curvature χ_0 and the sign k of the curvature:

$$D(\chi) = \begin{cases} \chi_0 \sin \chi/\chi_0 & k = 1 \\ \chi & k = 0 \\ \chi_0 \sinh \chi/\chi_0 & k = -1 \end{cases} \tag{2}$$

   The *redshift* z is defined by $1 + z = a^{-1}$, and is directly observable, while the time variable is not so easily measured. We can define the *angular diameter distance* $D_A(z) = D[\chi(z)]/(1 + z)$ to be the ratio of a transverse proper distance to the angle it subtends; and we can define the *lookback time* $T(z)$ to be the elapsed time from redshift z to the present. Using the fact that light travels on null geodesics with $ds = 0$, it's not hard to show that a light source observed to have redshift z has

$$\chi(z) = \int_{\text{then}}^{\text{now}} \frac{c\,dt}{a(t)} = \int_0^z \frac{c\,dz'}{H(z')} \tag{3}$$

$$T(z) = \int_{\text{then}}^{\text{now}} dt = \int_0^z \frac{dz'}{(1+z')H(z')}, \tag{4}$$

$$H(z) \equiv \frac{1}{a}\frac{da}{dt}. \tag{5}$$

Remember that we define $H_0 = H(z=0)$ at the present day, and it is common to write $H_0 = hH_{100}$, $H_{100} = 100\,\mathrm{km\,s^{-1}\,Mpc^{-1}}$. More generally I'll write $H(z) = h(z)H_{100}$.

If General Relativity is correct, the scale factor obeys the Friedmann equation

$$\frac{\dot{a}^2}{a^2} = \frac{8\pi G\rho}{3} - \frac{kc^2}{a^2\chi_0^2} + \frac{\Lambda}{3} \tag{6}$$

$$\Rightarrow h^2(z) = \omega_m(1+z)^3 + \omega_r(1+z)^4 + \omega_k(1+z)^2 + \omega_X f_X(z). \tag{7}$$

In the second line we have defined $\omega_m = \Omega_m h^2$, where Ω_m is the *present-day* ratio of the matter density to the critical density $\rho_{\rm crit} = 3H_0^2/8\pi G$. Similarly ω_r is the present-day radiation density; we've used the fact that the energy density of non-relativistic matter scales as a^{-3} while that of radiation as a^{-4}. $\omega_k = -kc^2/\chi_0^2 H_0^2$ defines the curvature, and ω_X and $f_X(z)$ describe the present-day energy density and evolution of the *dark energy*. Note that $f_X(z) = 1$ describes the effect of a cosmological constant Λ.

Finally, we're going to assume that we know the *equation of state* relation $P = w(a)\rho$ for the dark-energy, and that it obeys the usual thermodynamic relation

$$P\,dV = -d(\rho V) \quad \Rightarrow w(a)\rho\,d(a^3) = -d(\rho a^3). \tag{8}$$

You should be able to derive a relation between ρ and a from this in terms of w; plugging in $w = 0$ for matter or $w = 1/3$ for radiation should give the scalings we assumed in the Friedmann equation (7).

Let's assume—for no good reason, really—that our dark energy obeys some equation of state $w(a) = w_0 + w_a(1-a)$.

So finally, here is your assignment:

(a) Prove the last equality (between the two integrals) for equation (3), and likewise for equation (4).

(b) Solve the differential equation (8) analytically to give $f_X(z)$ for our chosen dark-energy model.

(c) Write a class called `Cosmology` which takes as constructor arguments the parameters

$$\{\omega_m, \omega_X, \omega_k, \omega_r, w_0, w_a\}. \tag{9}$$

The defaults are: $\{0.127, 0.406, 0, 0, -1, 0\}$, which give the "concordance model" from WMAP (ignoring the radiation). This class should have methods:

- `h(z)` which returns $H(z)$ in units of H_{100}.
- `DA(z)` which return $D_A(z)$ in units of $c/H_{100} = 2998$ Mpc.
- `T(z)` which returns the lookback time in units of $1/H_{100} = 9.8$ Gyr.
- `age()` which returns the time since the Big Bang, in the same units.

You will obviously have to do some integrations. Implement the method of your choice, using Numerical Recipes routines if you like. The values returned should be accurate to 1 part in 10^4 or better.

(d) The first-year WMAP results suggest that the angular-diameter distance to the recombination redshift $z = 1088$ is 13.7 ± 0.5 Gpc. For a Universe with a cosmological constant ($w_0 = -1, w_a = 0$), has no radiation, and the default matter and dark-energy distributions, what is the allowed range of ω_k?