Conductance fluctuations in small disordered conductors: Thin-lead and isolated geometries

R. A. Serota*
Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139

Shechao Feng
Schlumburger-Doll Research, Old Quarry Road, Ridgefield, Connecticut 06877-4108

C. Kane and P. A. Lee
Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts 02139
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We extend the analysis of conductance fluctuations in small disordered metallic systems beyond the conventional thick-lead geometry to thin-lead and isolated geometries. We find that, for the thin-lead geometry, the conductance fluctuations are still given by the "universal" value e^2/h, independent of the lead width. In the isolated geometry, the conductance fluctuation is enhanced by a factor $(L_{\rm in}/L)^2 \gg 1$ over e^2/h. The typical distance between consecutive peaks and valleys in the structure of conductance fluctuations, both as a function of external magnetic field and of chemical potential, is found to be dramatically reduced in both of these restrictive geometries.

Recently, the phenomenon of "universal fluctuations" in the metallic regime of small disordered conductors has attracted much attention in condensed-matter physics. Experimentally,[1,2] it was shown that the conductance has a sample-specific, but reproducible noiselike structure as a function of external magnetic field and chemical potential, at low temperatures. Theoretically,[3-5] this phenomenon was understood in terms of quantum interference between different multiple elastic scattering paths of the electron. Moreover, the average value of the conductance fluctuations (CF's) was established to be "universal"—of the order e^2/h—independent of the sample size or degree of disorder, as long as the conductor remains in the metallic regime.

So far, however, both theory and experiment have addressed only the thick-lead geometry in which the width t of the leads made of a clean metal is of the same order as the sample size L, $t \sim L$. This geometry is shown in Fig. 1(a). In this Brief Report we address the problem of CF's in thin-lead and isolated geometries, depicted in Figs. 1(b) and 1(c), respectively. Our interest in this problem was motivated by the fact that due to "entrapment" of diffusing electrons in such restrictive geometries, the fluctuations in the electronic level density (LD) at the Fermi level can be greatly enhanced. Since the CF is proportional to the fluctuation in the LD, one would expect an enhancement of the CF as well. Indeed, we demonstrate below that for the isolated geometry of Fig. 1(c), the CF is enhanced compared to the universal value e^2/h. On the other hand, it turns out that for the thin-lead geometry in Fig. 1(b) the enhancement of the fluctuation in LD is offset by the reduced conductance, so that CF is still given by e^2/h. In the following we first give a heuristic argument which produces all the results correctly. These results are then derived using diagrammatic techniques.

Let us first address the thick-lead geometry. We start with Ohm's law for the conductance $G=\sigma A/L$, where σ is the conductivity and $A \approx L^2$ is the cross-sectional area of the sample. Using the Einstein relation $\sigma = e^2 D\nu/\Omega$, where ν is the LD and Ω is the sample volume, we find for the dimensionless conductance $g \equiv G/(e^2/h) = \nu(hD/L^2)$. The sample is metallic when $g \gg 1$. Since $\tau_{\rm diff} = L^2/D$ is the time it takes for an electron to diffuse out of the sample (diffusion time), g can be rewritten as

$$g = \nu h/\tau_{\rm diff}\ , \tag{1}$$

a relation first given by Thouless.[6] Recently, Al'tshuler and Shklovskii[7] used Eq. (1) to give a simple explanation of the CF. They showed that the fluctuation in LD and the fluctuation in the diffusion coefficient give contributions of the same order of magnitude to CF, so that the rms CF is given by

$$\delta g \approx \delta\nu h/\tau_{\rm diff}\ . \tag{2}$$

They also showed that due to the phenomenon of "level repulsion," the fluctuation in LD is proportional to the diffusion time, $\delta\nu \approx \tau_{\rm diff}/h$. Substituting this into Eq. (2), one obtains the universal value $\delta g \approx 1$.

We can now examine the thin-lead geometry by generalizing the ideas in Refs. 6 and 7. First, it is plausible to assume that Eq. (1) applies to any geometry with leads. In the particular geometry of Fig. 1(b), diffusing electrons get effectively "trapped" inside the sample, as manifested by an enhancement of the diffusion time, $\tau'_{\rm diff} = (D/L^2)(L/t)^{d-1}$, where t^{d-1} $(t \ll L)$ is the cross-sectional area of the leads. Thus we conclude that in this case the conductance is $(t/L)^{d-1}$ times smaller than in the thick-lead case: $g' = \nu h/\tau'_{\rm diff} = \nu(hD/L^2)(t/L)^{d-1}$. Second, the argument of Ref. 7 can be modified to show that the fluctuation in LD is determined by the "level lifetime:" the lifetime of a coherent

quantum-mechanical state. In any geometry with leads, this time is the diffusion time. Thus $\delta\nu \approx \tau'_{\rm diff}/h$. From this and Eq. (2), we obtain that the CF in the thin-lead geometry has the universal value $\delta g \approx 1$, irrespective of the fact that the fluctuation in LD is enhanced compared to the thick-lead geometry due to an increase in the diffusion time. We thus observe that in any geometry with leads we should expect conductance fluctuations to have the universal value of the order of e^2/h, independent of the width of the leads. This conclusion is quite natural if we view the sample with the disordered leads [unshaded part in Fig. 1(b)] as a "black box" and consider the transmission probability through such a black box. The statement of universal conductance fluctuation is the same as stating that the quantity $\mathrm{Tr}(t^\dagger t)$, where t is the transmission matrix, has fluctuations of unity regardless of the nature of disorder.[5] Evidently it applies to the present geometry as well. Since conductance itself decreases with t, we expect that the sample would eventually cross over to the regime in which electrons are effectively localized inside the sample. Clearly, $\delta g \approx g \approx 1$ then.

Consider now the isolated geometry [Fig. 1(c)]. The conductance of the sample can be determined, for instance, by measuring the absorption coefficient of small

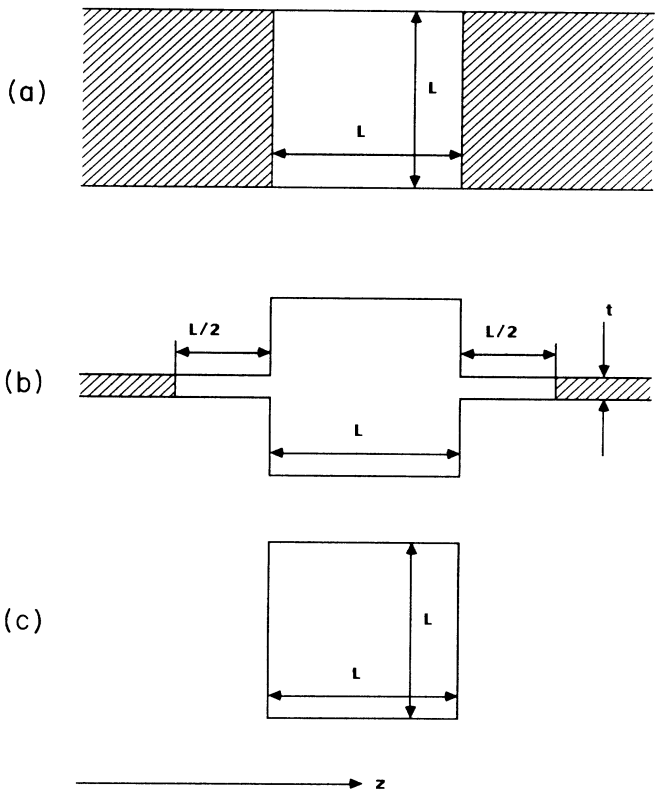

FIG. 1. Three different geometries for which conductance fluctuations are calculated: (a) thick-lead geometry of size L; (b) thin-lead geometry of size L and width $t \ll L$; (c) isolated geometry. Unshaded areas indicate regions of disordered conductor. Shaded areas indicate regions of clean metallic leads. Square (cubic) shapes are chosen for simplicity.

particles in a microwave cavity. Since the electrons cannot diffuse out of the sample, however, the level lifetime is no longer determined by D/L^2, but by the time between consecutive inelastic collisions, $\tau_{\rm in}$.[8] The fluctuation in LD is then given by $\delta\nu \approx \tau_{\rm in}/h$, which leads to the relation

$$\delta g \approx (D\tau_{\rm in}/L^2) = (L_{\rm in}/L)^2 , \tag{3}$$

where $L_{\rm in}$ is the inelastic diffusion length. Thus we observe that provided $L_{\rm in} \gg L$, conductance fluctuations in isolated samples can be dramatically enhanced over the universal value e^2/h. In fact, one can show that in the limit when fluctuations of LD are as large as LD itself,[7] conductance fluctuations can be as large as conductance itself, $\delta g \approx g \gg 1$.

As was mentioned in the Introduction, the aperiodic structure of conductance is observed experimentally[1,2] as a function of magnetic field or as a function of chemical potential. We now estimate the typical distance between consecutive peaks and valleys in such structures, i.e., the energy and magnetic field correlation range,[3] $E_{\rm corr}$ and $B_{\rm corr}$, respectively. At low temperatures the inverse level lifetime defines the energy range Γ within which the energy levels are correlated. Thus when $T < \Gamma$, we have $E_{\rm corr} \approx \Gamma$, whereas when $T > \Gamma$, one obtains[3] $E_{\rm corr} \approx T$. In geometries with leads the level lifetime is given by the diffusion time, thus we find that $\Gamma = h/\tau_{\rm diff}$, which yields $\Gamma = (hD/L^2)$ in the thick-lead case and $\Gamma = (hD/L^2)(t/L)^{d-1}$ in the thin-lead one. Consequently, we observe that (i) the onset of the universal value of conductance fluctuation occurs at a lower temperature, and (ii) the structure of the conductance as a function of chemical potential is more compact in the thin-lead geometry. The same pertains to the isolated geometry in the regime when $L > L_{\rm in}$, since here the level lifetime is defined by inelastic processes, so that $\Gamma = h/\tau_{\rm in} = (hD/L_{\rm in}^2)$.

To find $B_{\rm corr}$, it is constructive to use the physical picture of interference among Feynman paths. In the thick leads case this picture gives[3] the value of $B_{\rm corr}$ from the condition that the flux through the area between two typical interfering paths (of order L^2), is a flux quantum ϕ_0, i.e., $B_{\rm corr}L^2 \approx \phi_0 = hc/e$. In the thin leads geometry, the typical Feynman-path length is longer, $L_{\rm Feyn} \approx L^2(L/t)^{(d-1)}$ (measured in units of l), since it takes longer for an electron to diffuse out of the sample. Similarly in the isolated geometry $L_{\rm Feyn} \approx L_{\rm in}^2$, when $L_{\rm in} > L$. Knowing $L_{\rm Feyn}$, the field correlation range is found from the relation

$$B_{\rm corr}L^2(L_{\rm Feyn}/L^2)^{1/2} \approx \phi_0 . \tag{4}$$

Equation (4) is obtained by observing that a diffusion electron covers the area of the sample a number of times $L_{\rm Feyn}/L^2$ before it loses coherence. Each time two interfering paths acquire a phase difference of the order BL^2/ϕ_0. Accumulation of these phase differences with random signs gives a total phase difference of $BL^2/\phi_0(L_{\rm Feyn}/L^2)^{1/2}$, and hence Eq. (4). Thus we see that the field correlation range can be dramatically reduced in thin-lead and isolated geometries.

We now turn to the detailed analytic calculations. Starting with the relation $G=I/V=P/V^2$, where P is the power dissipation, we find by applying the Kubo formula that

$$\langle G\rangle = V^{-2}\int\int d\mathbf{r}\,d\mathbf{r}' E_\alpha(\mathbf{r})\langle\sigma_{\alpha\beta}(\mathbf{r}\mid\mathbf{r}')\rangle E_\beta(\mathbf{r}')\ , \tag{5}$$

where $\sigma_{\alpha\beta}(\mathbf{r}\mid\mathbf{r}')$ is the conductivity tensor and $\mathbf{E}(\mathbf{r})$ is the classical electric field, which satisfies the Maxwell equations. Using current conservation, Eq. (5) can be rewritten as

$$\langle G\rangle = L^{-2}\int\int d\mathbf{r}\,d\mathbf{r}'\langle\sigma_{zz}(\mathbf{r}\mid\mathbf{r}')\rangle\ . \tag{6}$$

To lowest order in the disorder parameter $1/k_F l$, there are two contributions to the averaged conductivity tensor: the short-range part [depicted in Fig. 2(a)], $\langle\sigma_{\alpha\beta}\rangle_{\rm SR}\sim\delta(\mathbf{r}-\mathbf{r}')$, and the long-range part [depicted in Fig. 2(b)], $\langle\sigma_{\alpha\beta}\rangle_{\rm LR}\sim\nabla_\alpha\nabla'_\beta P(\mathbf{r}\mid\mathbf{r}')$, where $P(\mathbf{r}\mid\mathbf{r}')$ is the diffusion propagator. It is then easily seen that $\langle\sigma_{\alpha\beta}\rangle=\langle\sigma_{\alpha\beta}\rangle_{\rm SR}+\langle\sigma_{\alpha\beta}\rangle_{\rm LR}$ satisfies the condition $\nabla_\alpha\langle\sigma_{\alpha\beta}\rangle=0$, as imposed by current conservation.

Equations (5) and (6) should obviously give the same answer for the average conductance. We can prove, however, that $\langle\sigma_{\alpha\beta}\rangle_{\rm LR}$ does not contribute to the integral in Eq. (5), and therefore in this equation only the short-range part of the conductivity tensor need be used. The integration is trivial for the thick-lead and isolated geometries due to the uniform electric field, and we immediately obtain the result of Eq. (1). For the thin-lead geometry we note that the potential drop occurs predominantly along the disordered parts of the leads [Fig. 1(c)], so that

$$\int d^d r\,\mathbf{E}^2(\mathbf{r})\approx(V/L)^{-2}(Lt^{d-1})\ , \tag{7}$$

Together with Eq. (5), this relation leads to the reduced value of g'.

We consider the CF:

$$\langle\delta G^2\rangle = V^{-4}\int\int\int\int d\mathbf{r}\,d\mathbf{r}'\,d\mathbf{r}_1\,d\mathbf{r}'_1[E_\alpha(\mathbf{r})E_\beta(\mathbf{r}')\langle\sigma_{\alpha\beta}(\mathbf{r}\mid\mathbf{r}')\sigma_{\gamma\delta}(\mathbf{r}_1\mid\mathbf{r}'_1)\rangle E_\gamma(\mathbf{r}_1)E_\delta(\mathbf{r}'_1)]\ . \tag{8}$$

As in the case of conductance, we can use the short-range part of the conductivity-tensor correlation function,[9] $\langle\sigma_{\alpha\beta}(\mathbf{r}\mid\mathbf{r}')\sigma_{\gamma\delta}(\mathbf{r}_1\mid\mathbf{r}'_1)\rangle$, which can be shown to be given, up to a constant of order unity, by the diagram[3] shown in Fig. 3. The diffusion propagator involved in this diagram satisfies the diffusion equation

$$(-\nabla^2+L_{\rm in}^{-2})P(\mathbf{r}\mid\mathbf{r}')=(hD)^{-1}\delta(\mathbf{r}-\mathbf{r}')\ , \tag{9}$$

subject to the boundary conditions that $P=0$ at the clean metal leads and $\nabla_n P=0$ on the insulating walls. $P(\mathbf{r}\mid\mathbf{r}')$ can then be expressed in terms of the eigenfunctions $\varphi_n(\mathbf{r})$ and eigenvalues λ_n of the diffusion equation as $P(\mathbf{r}\mid\mathbf{r}')=\sum_n\lambda_n^{-1}\varphi_n(\mathbf{r})\varphi_n(\mathbf{r}')$. This allows us to compute the diagram in Fig. 3, and the value of CF can then be derived from Eq. (8). For the case of a uniform electric field, which is true in the thick-lead and isolated geometries, we find

$$\langle\delta g^2\rangle=(hD/L^2)^2\sum_n\lambda_n^{-2}\approx(hD/L^2)\lambda_0^{-2}\ , \tag{10}$$

where the sum was approximated by the contribution from the lowest eigenvalue. For thick leads we have $\lambda_0=hD(1/L^2+1/L_{\rm in}^2)\approx hD/L^2$ at low temperatures, whereas for an isolated sample we have $\lambda_0=hD/L_{\rm in}^2$ independent of L. Substitution of the latter value of λ_0 in Eq. (10) yields the result of Eq. (3).

In the same manner, we obtain for the thin-leads geometry

$$\langle\delta g^2\rangle\approx(hD/L^2)^2\lambda_0^{-2}\left[V^{-4}L^{4-2d}\left[\int d\mathbf{r}\,\mathbf{E}^2(\mathbf{r})\right]^2\right]\ , \tag{11}$$

where

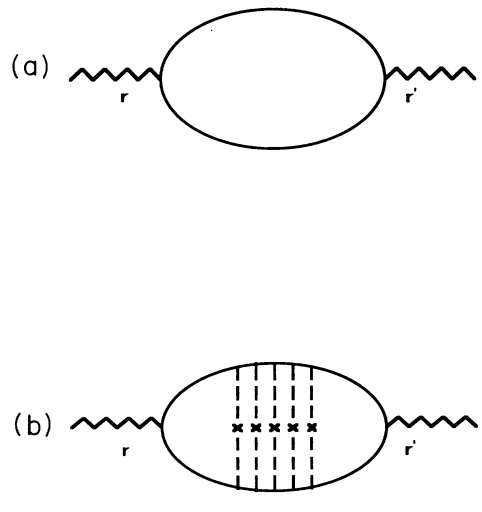

FIG. 2. Diagrams which contribute to the averaged conductivity tensor $\langle\sigma_{\alpha\beta}(\mathbf{r}\mid\mathbf{r}')\rangle$. (a) Short-range part $\langle\sigma_{\alpha\beta}\rangle_{\rm SR}\sim\delta(\mathbf{r}-\mathbf{r}')$; (b) long-range part $\langle\sigma_{\alpha\beta}\rangle_{\rm LR}\sim\nabla_\alpha\nabla'_\beta P(\mathbf{r}\mid\mathbf{r}')$. $P(\mathbf{r}\mid\mathbf{r}')$ is the diffusion propagator such that $\nabla^2 P(\mathbf{r}\mid\mathbf{r}')\sim-\delta(\mathbf{r}-\mathbf{r}')$.

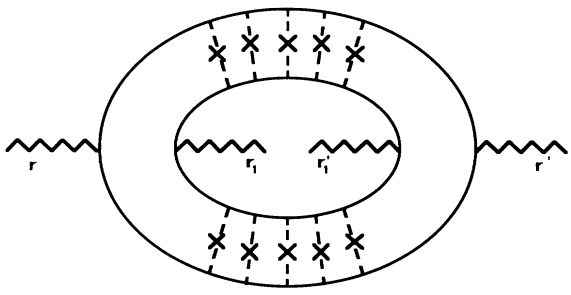

FIG. 3. One of the diagrams which contributes to the short-range part of the conductivity tensor correlation function $\langle\sigma_{\alpha\beta}(\mathbf{r}\mid\mathbf{r}')\sigma_{\gamma\delta}(\mathbf{r}_1\mid\mathbf{r}'_1)\rangle$.

$$\lambda_0 = hD[(1/L^2)(t/L)^{d-1} + 1/L_{\rm in}^2]$$
$$\cong hD(1/L^2)(t/L)^{d-1}$$

at low temperatures. Evidently, h/λ_0 describes the enhancement of the diffusion time and LD in the thin-lead geometry. However, with the help of Eq. (7) we find that this enhancement is offset by the electric field factor which is also ultimately responsible for the reduced conductance. As a result we arrive at the universal value $\langle \delta g^2 \rangle \approx 1$ of CF.

In the presence of a small field difference $\Delta\beta$ between the loops of the diagram in Fig. 3, the diffusion equation has to be modified by substituting $\nabla \to \nabla - ie\Delta A$. The perturbation expansion in ΔB yields the following change for the lowest eigenvalue:

$$\lambda_0 \to \lambda_0 + (hD/L^2)(\Delta B L^2/\phi_0)^2 .$$

Using this relation, we find the field correlation range from the relation $B_{\rm corr}L^2(hD/\lambda_0 L^2)^{1/2} = \phi_0$, which reproduces the desired results. The energy correlation range $E_{\rm corr}$ can be obtained similarly, and the result coincides with the qualitative estimate given above.

We can extend the above analysis to obtain the conductance fluctuation due to moving impurities.[10,11] In Ref. 11 it was shown that for the thick-lead geometry of Fig. 1(a), the conductance change due to moving one impurity atom (by a distance $\delta r > 1/k_F$) in the system is given by

$$\delta g_1^2 \approx (k_F 1)^{-1}(k_F L)^{-(d-2)} \approx 1/g < 1 .$$

If m impurities move in an uncorrelated fashion, then $\delta g^2 = x$, when $x \equiv m\delta g_1^2 < 1$, and δg^2 saturates at the universal value ≈ 1 when $x > 1$. For the thin-lead geometry of Fig. 1(b), a parallel calculation yields an enhanced conductance change due to moving impurities. In particular, we have $x' = m\delta g_1^2 (L/t)^{d-1}$. Notice, however, that one can still write the conductance fluctuation due to moving one impurity as $\delta g_1'^2 \approx 1/g' < 1$, since the conductance in the thin-lead case is reduced by the same factor $(L/t)^{d-1}$. Similar calculation for the isolated geometry gives $x'' = m\delta g_1^2 (L_{\rm in}/L)^2$.

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*Present address: Department of Physics, University of Cincinnati, Cincinnati, Ohio 45221.

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[8]In Ref. 7 it was not recognized that in the thin-lead and isolated geometries the level repulsion mechansim should be applied to the bands of the width $h/\tau_{\rm diff}'$ and $h/\tau_{\rm in}$, respectively, and not to the band of the width $h/\tau_{\rm diff}$, as is the case in the thick-lead geometry. This led Ref. 7 to conclude that $\delta g \approx 1$ for isolated samples, which we believe to be incorrect.

[9]The detailed discussion of the long-range part of the conductivity correlation function will be presented in a separate publication.

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