Voltage fluctuations in multilead devices

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The problem of voltage fluctuations in multilead devices can be formulated either in terms of fluctuating electric fields or fluctuating transmission amplitudes. We point out some subtleties regarding current conservation in the fluctuating-electric-field approach and then show that by treating the fluctuations in the transmission amplitudes exactly, the two approaches can be shown to be identical. We show that in the presence of a magnetic field the antisymmetric part of the conductivity has a finite divergence; however, when evaluated in perturbation theory, its effect is of higher order in the disorder and can be ignored. Finally, we present some results for voltage fluctuations in three-, four-, and six-lead devices.

I. INTRODUCTION

The phenomenon of universal conductance fluctuations has attracted considerable attention in the past few years. Experiment[1] and theory[2,3] have shown that the conductance of small disordered conductors displays reproducible aperiodic fluctuations of the order of e^2/h as a function of chemical potential or magnetic field, independent of the sample size or degree of disorder. These are quantum-interference effects which are a result of the long-range phase coherence of electrons on length scales smaller than the inelastic scattering length, $L_{\rm in}$.

The original conductance-fluctuation calculations[2,3] were carried out for a two-lead geometry in which the voltage is measured between the same leads through which the current is passed. In that case the conductance has been shown to fluctuate "universally" when the sample dimensions are smaller than $L_{\rm in}$. Conductance is often measured using a four-lead geometry with distinct current and voltage leads. The voltage is measured at the ends of the voltage leads which carry no current. Classically, one expects that such measurements should probe the voltages at the points of attachment of the voltage leads, and that the current divided by this voltage is the conductance of the segment of conductor between the voltage leads. Recent experiments,[4,5] however, have brought into question the role of voltage leads when measuring quantum fluctuations in small devices. In particular, it has been shown that for a fixed current the fluctuations in the voltage measured between the voltage leads approach a finite value as the leads are moved closer together, even though the average voltage goes to zero. According to classical expectations, this would seem to imply that the fluctuations in the conductance diverge as the leads are moved closer together, in violation of the theory of universal conductance fluctuations.

The problem with this interpretation is that our classical expectations are based on the assumption that the leads can be considered as independent resistors connected to the main body of the sample, so that when there is no current in a voltage lead, there is no voltage across it. When considering quantum fluctuations, one must keep in mind that the electrons are phase coherent throughout the entire sample, including the leads up to a length $L_{\rm in}$. The leads cannot be treated as if they were independent of the rest of the sample. The electrical conductivity, $\sigma(\mathbf{r},\mathbf{r}')$, which relates the current density at $\mathbf{r}$ to the electric field at $\mathbf{r}'$, is highly nonlocal. The current as $\mathbf{r}$ depends on the electric field throughout the entire sample. Therefore, there can be an electric field in a voltage probe even when there is no current there,[6] and the voltage measured at the end of the voltage lead need not be the same as the voltage at the point of attachment. It is thus clear that the conductance measured on a four-lead device is not the same as the conductance which is addressed in the theory of universal conductance fluctuations. The voltage leads strongly affect the results.

In order to understand such measurements and to address the question of whether there is anything universal in the fluctuations observed, it is necessary to have a theory which treats the entire sample, including the leads, on an equal footing. The current and voltage leads are assumed to be connected to large reservoirs in which the electrons come to thermal equilibrium by inelastically scattering. Electrons can enter or leave the leads; however, when they leave they are not expected to return phase-coherently.[7] The problem can be specified by fixing the current passing through each lead to be zero in the voltage leads and a constant I in the current leads. We would then like to calculate the fluctuations in the voltage measured between the ends of the voltage leads.

This problem has been approached from two different angles, which emphasize different physical principles. Maekawa *et al.*[6] have emphasized the nonlocality of the electrical conductivity $\sigma(\mathbf{r},\mathbf{r}')$. By imposing the fact that the current in the sample is fixed, they calculate the fluctuation in the electric field, δE, necessary to counterbalance the nonlocal fluctuation in the conductivity, $\delta\sigma(r,r')$. By integrating δE between the ends of the volt-

age leads they are able to calculate the voltage fluctuations. This method stresses the fact that there are fluctuating electric fields present in the voltage leads, so that much of the fluctuations in the voltage measured are actually coming from the leads.

Büttiker[8] has considered fluctuations in the transmission amplitudes, G_{ij}, for an electron to propagate from lead i to lead j. From simple arguments similar to those used to explain conductance fluctuations in two-lead devices,[9] the transmission amplitudes are expected to fluctuate universally by an amount of the order e^2/h. In Sec. III we calculate these fluctuations explicitly and describe the sense in which they are universal. These transmission amplitudes can be identified with a conductance matrix which relates the currents to the voltages at each lead,[10–12]

$$I_i = \sum_j G_{ij} V_j \ . \tag{1.1}$$

By inverting this expression, the voltages at each lead can be expressed in terms of the currents at each lead and the transmission amplitudes. The voltage fluctuations can then be calculated with a knowledge of the fluctuations in G_{ij}. This method emphasizes the fact that the transmission amplitudes do fluctuate "universally," but that they are not measured directly in an experiment on a four-lead device. The voltage measured is a somewhat complicated function of the different transmission amplitudes.[8] In a two-lead device, the conductance measured is precisely the transmission amplitude,[11] so it does fluctuate universally. As will be shown below, it should be possible to extract the transmission amplitudes from current and voltage measurements on multilead devices.

In Secs. II and III we take a closer look at these two approaches. In Sec. II we point out some subtleties regarding the nonlocality of the electrical conductivity in the formulation of Maekawa *et al.*[6] We then show in Sec. III that if one explicitly calculates the fluctuations in the transmission amplitudes, then the two methods give identical results. In Sec. IV we discuss the role of a magnetic field in these calculations and point out some complications which arise when treating the problem of magnetic field antisymmetry. Finally, in Sec. V we show some results for three-, four-, and six-lead geometries and discuss their implications.

II. THE FLUCTUATING-ELECTRIC-FIELD APPROACH

In this section we describe the formulation presented in Ref. 6 and point out some subtleties that arise. The problems have to do with the fact that the ensemble-averaged conductivity, $\langle \sigma_{\alpha\beta}(\mathbf{r},\mathbf{r}') \rangle$, has been shown to be nonlocal as a consequence of current conservation.[10] Similar nonlocal terms are present in the correlation function of the conductivity. This nonlocality has nothing to do with quantum interference; its origin has to do with the classical requirement of current conservation. It turns out that if one interprets the electric field as a "classical" field in a sense to be described below, these nonlocal corrections can be shown to disappear, so that the results in Ref. 6, in which these nonlocal corrections are ignored, are unaffected.

Maekawa *et al.*[6] treat the problem of calculating the voltage fluctuations in a sample which is composed of quasi-one-dimensional segments, whose width is much smaller than their length. Their approach is to enforce the fact that the total current passing through any given cross section is fixed and does not fluctuate. They therefore write

$$\delta I = \delta \int d\mathbf{S}_\alpha d\mathbf{r}' \sigma_{\alpha\beta}(\mathbf{r},\mathbf{r}') E_\beta(\mathbf{r}') = 0 \ , \tag{2.1}$$

where the integral over $d\mathbf{S}_\alpha$ is an integral over any cross section. This is then expanded about the averaged quantities,

$$\int d\mathbf{S}_\alpha d\mathbf{r}' \langle \sigma_{\alpha\beta}(\mathbf{r},\mathbf{r}') \rangle \delta E_\beta(\mathbf{r}') + \delta\sigma_{\alpha\beta}(\mathbf{r},\mathbf{r}') \langle E_\beta(\mathbf{r}') \rangle = 0 \ . \tag{2.2}$$

Maekawa *et al.* then assumed that $\langle \sigma_{\alpha\beta}(\mathbf{r},\mathbf{r}') \rangle = \sigma_0 \delta(\mathbf{r}-\mathbf{r}')$, which allowed them to express δE averaged over a cross section in terms of $\delta\sigma$ and $\langle E \rangle$. However, it has recently been shown that the averaged conductivity is actually long ranged,[10]

$$\begin{aligned} \langle \sigma_{\alpha\beta}(\mathbf{r},\mathbf{r}') \rangle &= \sigma_0 [\delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}') - \nabla_\alpha \nabla'_\beta d(\mathbf{r},\mathbf{r}')] \\ &\equiv \sigma_0 \phi_{\alpha\beta}(\mathbf{r},\mathbf{r}') \ , \end{aligned} \tag{2.3}$$

where $d(\mathbf{r},\mathbf{r}')$ is the rescaled diffusion propagator satisfying the equation $-\nabla^2 d(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}')$ subject to the boundary conditions that $d(\mathbf{r},\mathbf{r}')=0$ on a conducting boundary and $\nabla_n d(\mathbf{r},\mathbf{r}')=0$ on an insulating boundary.[2,3] It is necessary for the conductivity to have such a form in order to satisfy the constraint of current conservation, $\nabla_\alpha \sigma_{\alpha\beta}(\mathbf{r},\mathbf{r}')=0$, which can be proven directly from the Kubo formula.[10] It therefore appears impossible to isolate δE in Eq. (2.2).

This problem can be avoided by introducing a classical electric field in the following way. In Ref. 10 it is shown that the constraint of current conservation on the electrical conductivity implies that the current at a given point depends only on the voltages at the leads. That is, it is independent of the precise electric field configuration. The true electric field may be a complicated function of position due to local charge imbalances. However, we can express the current in terms of the simple classical electric field which would exist if there was no charge. The classical field satisfies $\nabla_\alpha E_\alpha^{\rm cl}=0$ subject to the boundary conditions that its normal component vanish on insulating boundaries and that it integrates to give the correct values of the voltage at the leads. The classical electric field will resemble the true electric field on length scales longer than the screening length. If the sample is composed of segments whose width is much smaller than their length than the classical field will be uniform in each segment except in the vicinity of a junction.

One can use the classical field in Eq. (2.2). The term involving the long-range part of the conductivity can be shown to vanish by integrating it by parts and using the fact that $\nabla_\alpha E_\alpha^{\rm cl}=0$ along with the boundary conditions satisfied by $d(\mathbf{r},\mathbf{r}')$. Therefore we can write

$$\delta E^{\rm cl}(\mathbf{r}) = -\frac{1}{\sigma_0}\int d\mathbf{r}'\delta\sigma(\mathbf{r},\mathbf{r}')E_0^{\rm cl}(\mathbf{r}') , \tag{2.4}$$

where since we are considering a sample composed of quasi-one-dimensional segments, we can remove the cross-sectional integrals and vector indices. Voltage fluctuations can be found by integrating $\delta E^{\rm cl}$ between the voltage leads. Provided the junctions are of negligible size, this integration can be carried out by doing a volume integral over the segments between the leads. Furthermore, $E_0^{\rm cl}$ will only be nonzero on the path between the current leads, where it will be equal to $I/\sigma_0 A$. We can therefore express the $\mathbf{r}'$ integral as an integral between the current leads. If the voltage is measured between leads a and b and the current is passed between leads c and d (see Fig. 1), then we can write (in the notation of Ref. 4)

$$\delta V_{ab,cd} = \frac{I}{\sigma_0^2}\int_a^b d\mathbf{r}\int_c^d d\mathbf{r}'\delta\sigma(\mathbf{r},\mathbf{r}') . \tag{2.5}$$

This form emphasizes the Onsager symmetry between the current leads and the voltage leads in the absence of a magnetic field.

To calculate the fluctuations, $\langle \delta V_{ab,cd}^2 \rangle$, one needs to calculate the quantity, $\langle \delta\sigma(\mathbf{r}_1,\mathbf{r}_2)\delta\sigma(\mathbf{r}_3,\mathbf{r}_4)\rangle$ using perturbation theory. The procedure is similar to the calculation of conductance fluctuations;[2,3] however, if we are not integrating over the external positions, it is necessary to include diagrams in which the external current vertices are dressed by diffusion ladders. There are many diagrams, and straightforward enumeration and evaluation is rather difficult; however, it has recently been shown that by exploiting the constraint of current conservation one can express the result as[10]

$$\langle \delta\sigma_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2)\delta\sigma_{\gamma\delta}(\mathbf{r}_3,\mathbf{r}_4)\rangle = \int d\mathbf{r}_1'\int d\mathbf{r}_2'\int d\mathbf{r}_3'd\mathbf{r}_4'\phi_{\alpha\alpha'}(\mathbf{r}_1,\mathbf{r}_1')\phi_{\beta\beta'}(\mathbf{r}_3,\mathbf{r}_2')\phi_{\gamma\gamma'}(\mathbf{r}_3,\mathbf{r}_3')\phi_{\delta\delta'}(\mathbf{r}_4,\mathbf{r}_4')\Gamma_{\alpha'\beta'\gamma'\delta'}(\mathbf{r}_1',\mathbf{r}_2',\mathbf{r}_3',\mathbf{r}_4') . \tag{2.6}$$

$\phi_{\alpha\beta}(\mathbf{r},\mathbf{r}') \equiv \delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}') - \nabla_\alpha\nabla_\beta' d(\mathbf{r},\mathbf{r}')$ is the function introduced earlier that has exactly the same form as the ensemble-averaged conductivity. $\Gamma_{\alpha\beta\gamma\delta}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)$ is the sum of all diagrams in which the impurity dressings of the current vertices are omitted. There are precisely the diagrams considered by Lee and Stone,[2] except that the external vertices are not integrated over. Diagrams in which the sign of the energy does not change at the current vertices have been shown to cancel,[3,10] so Γ can be expressed in terms of the two diffusion diagrams shown in Fig. 2. Their evaluation leads to[10]

$$\Gamma_{\alpha\beta\gamma\delta}(\mathbf{r},\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4) = 16\left[\frac{e^2}{h}\right]^2 [\delta_{\alpha\beta}\delta_{\gamma\delta}\delta(\mathbf{r}_1-\mathbf{r}_2)\delta(\mathbf{r}_3-\mathbf{r}_4)d(\mathbf{r}_1,\mathbf{r}_3)^2 + \delta_{\alpha\gamma}\delta_{\beta\delta}\delta(\mathbf{r}_1-\mathbf{r}_3)\delta(\mathbf{r}_2-\mathbf{r}_4)d(\mathbf{r}_1,\mathbf{r}_2)^2 + \delta_{\alpha\delta}\delta_{\beta\gamma}\delta(\mathbf{r}_1-\mathbf{r}_4)\delta(\mathbf{r}_2-\mathbf{r}_3)d(\mathbf{r}_1,\mathbf{r}_2)^2] . \tag{2.7}$$

[Note that diagrams 2(a) and 2(b) and similar diagrams add to give the first term in (2.7)].

Inelastic effects can be included by putting an inelastic cutoff in the diffusion propagators contained in Γ, so that they satisfy $(-\nabla^2 + L_{\rm in}^{-2})d(\mathbf{r},\mathbf{r}') = \delta(\mathbf{r}-\mathbf{r}')$ subject to the same boundary conditions.[2,3] This cutoff appears because these diffusion propagators represent ladder diagrams between Green's functions which represent different measurements. Therefore they should be connected only by impurity lines and not by interaction lines. Thus the usual cancellation between self-energy and vertex corrections does not occur.[2] The inelastic cutoff should not be included in the diffusion propagators which dress the current vertices and appear in $\phi_{\alpha\beta}(\mathbf{r},\mathbf{r}')$. These ladders are between Green's functions which describe the same

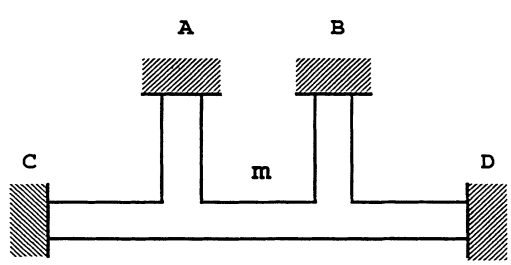

FIG. 1. Typical four-lead geometry in which each lead is connected to a large reservoir where electrons come to thermal equilibrium.

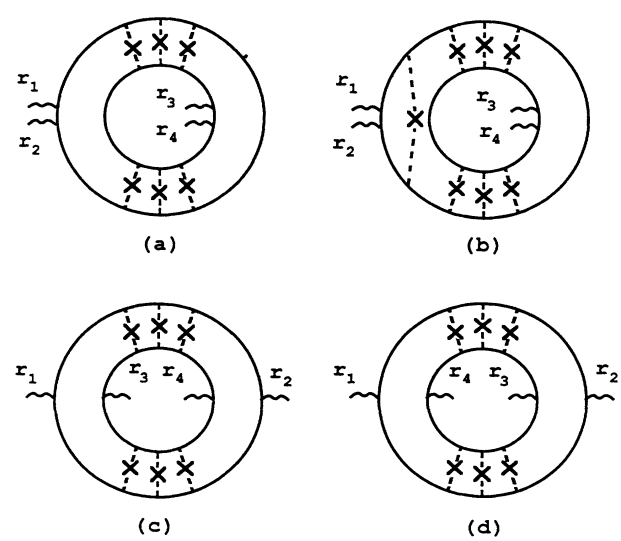

FIG. 2. Diagrams contributing to $\Gamma_{\alpha\beta\gamma\delta}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)$.

measurement, so the cancellation mentioned above does occur. The form of $\phi_{\alpha\beta}(\mathbf{r},\mathbf{r}')$ is a consequence of particle conservation and cannot be altered by inelastic scattering.

The quantity $\langle \delta V_{ab,cd}^2 \rangle$ is calculated by integrating $\langle \delta\sigma\delta\sigma \rangle$ over all of its variables between either leads a and b or leads c and d. The terms which involve the long-range part of $\phi_{\alpha\beta}(\mathbf{r},\mathbf{r}')$ involve integrals of $\nabla_\alpha \nabla'_\beta d(\mathbf{r},\mathbf{r}')$ which vanish because of the boundary condition obeyed by d. Therefore, the final expression can be written as

$$\langle \delta V_{ab,cd}^2 \rangle = \frac{I^2}{\sigma_0^4} \int_a^b d\mathbf{r}_1 \int_c^d d\mathbf{r}_2 \int_a^b d\mathbf{r}_3 \int_c^d d\mathbf{r}_4 \Gamma(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4) \ . \tag{2.8}$$

The indices in $\Gamma_{\alpha\beta\gamma\delta}$ are all taken to be in the direction of the segments of wire. Equation (2.8) is valid for any sample that is composed of quasi-one-dimensional segments in which the effects of the junctions are negligible, provided it is singly connected in the sense that there is a single well-defined path between any two points (i.e., no loops). Otherwise the current could fluctuate and (2.1) would not be valid.

We see from Eq. (2.8) that the long-range terms do not contribute after integration. Equation (2.8) is the formula used by Maekawa *et al.*,[6] and we therefore conclude that their result is correct. If one wished to know δE, however, it would be necessary to keep the long-range terms in Eq. (2.4). For instance, Γ has a finite divergence, whereas $\delta E^{\rm cl}$ cannot. These long-range terms cancel whenever their arguments are integrated. A similar cancellation occurred in Ref. 13. This will not be the case when we treat the problem in terms of transmission amplitudes below. The long-range terms represent a nonlocality that is purely classical in origin, since it is a consequence of current conservation.[9] This nonlocality extends throughout the entire sample and is independent of the inelastic scattering length. This is to be contrasted with the nonlocal correlations that are contained in Γ, which arises from quantum interference and are responsible for conductance fluctuations. These correlations are nonlocal only on lengths shorter than the inelastic length.

III. FLUCTUATING-TRANSMISSION-AMPLITUDE APPROACH

We now proceed to discuss Büttiker's approach[8] and show that if one explicitly calculates the correlations in the transmission amplitudes, then the result is identically Eq. (2.8). It has been shown that the current in each lead can be related to the voltage at each lead via the conductance matrix,[10,12]

$$I_i = \sum_j G_{ij} V_j \ , \tag{3.1}$$

where G_{ij} can be expressed as an integral of the electrical conductivity over the cross sections of leads i and j,

$$G_{ij} = \int d\mathbf{S}_{i\alpha} \int d\mathbf{S}'_{j\beta} \sigma_{\alpha\beta}(\mathbf{r},\mathbf{r}') \ . \tag{3.2}$$

The elements of this conductance matrix can be interpreted as transmission amplitudes for electrons to go from lead i to lead j.[10,11]

In this problem we would like to express the voltages at each lead in terms of the currents at each lead, so we should define a resistance matrix R_{ij} such that

$$V_i = \sum_j R_{ij} I_j \ . \tag{3.3}$$

G_{ij} and R_{ij} are not strictly matrix inverses of each other, since R_{ij} is only defined up to an additive constant; however, they do satisfy

$$\sum_k (R_{ak} - R_{bk}) G_{ki} = \sum_k G_{ik} (R_{ka} - R_{kb}) = \delta_{ai} - \delta_{bi} \ . \tag{3.4}$$

We would like to calculate fluctuations in the quantity

$$V_{ab,cd} = I[(R_{ac} - R_{ad}) - (R_{bc} - R_{bd})] \ . \tag{3.5}$$

The fluctuations in the resistance matrix can be calculated to linear order by taking the variation of Eq. (3.4), and one finds

$$\delta V_{ab,cd} = -I \sum_{l,m} (R_{al} - R_{bl}) \delta G_{lm} (R_{mc} - R_{md}) \ . \tag{3.6}$$

To evaluate $\langle \delta V_{ab,cd}^2 \rangle$ it is necessary to know the correlations in the different transmission amplitudes. Büttiker[8] has argued that the transmission amplitudes should fluctuate by an amount of the order e^2/h and that different transmission amplitudes should be uncorrelated. His arguments, however, are insensitive to the geometrical dependence of the fluctuations on the ratios of the sample dimensions. He assumes that the transmission-amplitude fluctuations are constants, so that the geometrical dependence in the voltage fluctuations comes entirely from the average resistance in (3.6). This assumption is not strictly correct, and we shall show how to obtain the exact geometry dependence of the correlations in the transmission amplitudes.

These correlations can be explicitly evaluated by expressing the transmission amplitude in terms of the conductivity as in Eq. (3.2). From (2.6) we get

$$\langle \delta G_{ij} \delta G_{kl} \rangle = \int d\mathbf{S}_{i\alpha} \int d\mathbf{S}_{j\beta} \int d\mathbf{S}_{k\gamma} \int d\mathbf{S}_{l\delta} \int d\mathbf{r}'_1 \int d\mathbf{r}'_2 \int d\mathbf{r}'_3 \int d\mathbf{r}'_4 \phi_{\alpha\alpha'}(\mathbf{r}_1,\mathbf{r}'_1) \phi_{\beta\beta'}(\mathbf{r}_2,\mathbf{r}'_2) \phi_{\gamma\gamma'}(\mathbf{r}_3,\mathbf{r}'_3) \phi_{\delta\delta'}(\mathbf{r}_4,\mathbf{r}'_4) \times \Gamma_{\alpha'\beta'\gamma'\delta'}(\mathbf{r}'_1,\mathbf{r}'_2,\mathbf{r}'_3,\mathbf{r}'_4) \ . \tag{3.7}$$

The integral over dS_i is an integration over the cross section of the ith lead. Equation (3.7) is completely general, and it applies to any geometry. The only input to the calculation is the diffusion propagator, which is the solution of the classical diffusion equation in that geometry. The resistance matrix can be calculated from knowledge of

$\langle G_{ij} \rangle$, and voltage fluctuations can then be calculated via (3.6).

In the case of a sample made up of quasi-one-dimensional segments the diffusion propagator is particularly simple, and the function $\phi(\mathbf{r},\mathbf{r}')$ is just a constant in each segment. Then, using Eq. (2.6), we can write

$$\langle \delta G_{ij} \delta G_{kl} \rangle = \sum_{i',j',k',l'} \phi_{ii'}\phi_{jj'}\phi_{kk'}\phi_{ll'} \int d\mathbf{r}_{i'} \int d\mathbf{r}_{j'} \int d\mathbf{r}_{k'} \int d\mathbf{r}_{l'} \Gamma(\mathbf{r}_{i'},\mathbf{r}_{j'},\mathbf{r}_{k'},\mathbf{r}_{l'}) \ . \tag{3.8}$$

The indices $i', \ldots, l'$ label the different one-dimensional segments of the sample which include internal segments as well as leads. The integrals over $d\mathbf{r}_{i'}$ are integrations in the i' segment. $\phi_{ii'}$ is the constant $\phi(\mathbf{r}_i,\mathbf{r}_{i'})$ integrated over the cross sections.

In order to simplify the calculation of voltage fluctuations and to make the connection with the result of the fluctuating electric field formulation, it is convenient to note that $\langle G_{ij} \rangle = \sigma_0 \phi_{ij}$, so that it would be tempting to express the $\phi_{ii'}$ in (3.8) in terms of $\langle G_{ij} \rangle$. The only problem is that i' can refer to internal segments of the sample. We shall show in Appendix A that the internal segments can be eliminated from the sum by changing the region of integration of Γ,

$$\langle \delta G_{ij} \delta G_{kl} \rangle = \frac{1}{\sigma_0^4} \sum_{i',j',k',l'} \langle G_{ii'} \rangle \langle G_{jj'} \rangle \langle G_{kk'} \rangle \langle G_{ll'} \rangle \int_x^{i'} d\mathbf{r}_1 \int_x^{j'} d\mathbf{r}_2 \int_x^{k'} d\mathbf{r}_3 \int_x^{l'} d\mathbf{r}_4 \Gamma(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4) \ , \tag{3.9}$$

where now $i', \ldots, l'$ are summed only over the leads and x is any intermediate point in the sample. This expression can then be inserted into Eq.(3.6) and the G's and R's cancel each other because of Eq. (3.4). The end result is

$$\langle \delta V_{ab,cd}^2 \rangle = \frac{I^2}{\sigma_0^4} \int_a^b d\mathbf{r}_1 \int_c^d d\mathbf{r}_2 \int_a^b d\mathbf{r}_3 \int_c^d d\mathbf{r}_4 \Gamma(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4) \ , \tag{3.10}$$

in exact agreement with the previous result, (2.8).

The fluctuating-electric-field approach[6] to voltage fluctuations is somewhat more direct than the fluctuating-transmission-amplitude approach,[8] and it may be argued that it is more fundamental based on the structure of the calculations. However, the simple result (3.10) is specific to the particular problem of a sample composed of quasi-one-dimensional leads. In more general geometries it is necessary to account for the nontrivial structure of the classical field, so the simplicity of (3.10) is lost. The fluctuating-transmission-amplitude approach is general and is capable of treating any geometry. In addition, the transmission-amplitude approach provides a bridge between multiprobe voltage fluctuations and the theory of universal conductance fluctuations.[2,3] The fluctuations in the transmission amplitudes are universal in the sense that they are of the order e^2/h and independent of the overall scale size (for $L < L_{\text{in}}$) and the degree of disorder. The measured conductance in a two-lead device fluctuates universally because it is precisely the transmission amplitude. The origin of the fluctuations in the transmission amplitudes can be understood in terms of the "diffusive" interference of electrons when they travel from one lead to another.[10]

It should be possible to extract the transmission amplitudes from voltage measurements by hooking up the current and voltage leads in different combinations. Consider an n-lead device. If there is time-reversal invariance, then there will be $n(n-1)/2$ independent transmission amplitudes (the diagonal ones can be expressed in terms of the off-diagonal ones by current conservation). The number of independent voltage measurements is constrained by voltage additivity and the Onsager symmetry[12] between current and voltage probes. They can be counted by fixing one of the current leads and one of the voltage leads to be at a single lead, and then counting the number of ways of hooking up the other two in the remaining $n-1$ leads, keeping in mind the Onsager symmetry. There will be $(n-1)(n-2)/2$ ways of hooking up the remaining two at different leads and $n-1$ ways of hooking them up at the same lead, so there will be a total of $n(n-1)/2$ independent measurements. These include measurements in which the current and voltage leads coincide. In order to extract the transmission amplitudes, it must be be possible to measure the voltage at the current leads. In Appendix B we will show explicitly how to extract the transmission amplitudes from voltage measurements on a three-lead device.

IV. EFFECT OF A MAGNETIC FIELD

The calculations presented up to this point in this paper were carried out under the assumption of time-reversal invariance, so they, strictly speaking, do not apply in the presence of a magnetic field. In this section we show that these results do apply to the symmetric part of the conductivity. In a recent paper, Isawa *et al.*[14] have extended the theory to explain the observed magnetic field asymmetry.[4,15] We point out some additional subtleties regarding current conservation, but show that their results are valid anyway.

To treat systems without time-reversal symmetry, it is necessary to use Streda's generalization of the Kubo formula,[16] which has previously been used to calculate fluctuations in the Hall conductivity.[16]

$$\sigma_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2)=\frac{e^2}{4\pi}\{G^+(\mathbf{r}_1,\mathbf{r}_2)v_{1\alpha}v_{2\beta}[G^+(\mathbf{r}_2,\mathbf{r}_1)-G^-(\mathbf{r}_2,\mathbf{r}_1)]-[G^+(\mathbf{r},\mathbf{r}_2)-G^-(\mathbf{r}_1,\mathbf{r}_2)]v_{1\alpha}v_{2\beta}G^-(\mathbf{r}_2,\mathbf{r}_1)\}\ , \tag{4.1}$$

where $v_{1\alpha}=[(\vec{\nabla}_{1\alpha}-\overleftarrow{\nabla}_{l\alpha})/2i-eA_\alpha(\mathbf{r}_1)/c]/m$. We ignore the term $ec\partial(G^+-G^-)/\partial B$ because it is too small at the magnetic fields of interest. In the presence of a magnetic field, $\sigma_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2,H)$ obeys the Onsager symmetry relation $\sigma_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2,H)=\sigma_{\alpha\beta}(\mathbf{r}_2,\mathbf{r}_1,-H)$. We can then decompose the conductivity into parts symmetric and antisymmetric with respect to the reversal of the magnetic field by writing

$$\sigma^{S,A}_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2)\equiv\tfrac{1}{2}[\sigma_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2,H)\pm\sigma_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2,-H)]=\tfrac{1}{2}[\sigma_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2,H)\pm\sigma_{\alpha\beta}(\mathbf{r}_2,\mathbf{r}_1,H)]\ .$$

From (4.1) we can then write

$$\begin{aligned}\sigma^S_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2)&=\frac{e^2}{4\pi}[G^+(\mathbf{r}_1,\mathbf{r}_2)-G^-(\mathbf{r}_1,\mathbf{r}_2)]v_{1\alpha}v_{2\beta}[G^+(\mathbf{r}_2,\mathbf{r}_1)-G^-(\mathbf{r}_2,\mathbf{r}_1)]\ ,\\ \sigma^A_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2)&=\frac{e^2}{4\pi}[G^+(\mathbf{r}_1,\mathbf{r}_2)v_{1\alpha}v_{2\beta}G^-(\mathbf{r}_2,\mathbf{r}_1)-G^-(\mathbf{r}_1,\mathbf{r}_2)v_{1\alpha}v_{2\beta}G^+(\mathbf{r}_2,\mathbf{r}_1)]\ .\end{aligned} \tag{4.2}$$

The symmetric part looks exactly like the Kubo formula, and it can be shown explicitly that $\nabla_\alpha\sigma^S_{\alpha\beta}=0$.[10] Therefore, all of the techniques presented in Secs. II and III are valid. The only effect of a magnetic field will be to eliminate the contribution from diffusion ladders in the particle-particle channel,[2,3,14] which will effectively cut the fluctuations in half.

The situation is more complicated for the antisymmetric part. From the equation of motion satisfied by the Green's functions, we find that

$$\nabla_\alpha\sigma^A_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2)=\frac{e^2}{2\pi i}\{[G^+(\mathbf{r}_1,\mathbf{r}_2)-G^-(\mathbf{r}_1,\mathbf{r}_2)]v_{2\beta}\delta(\mathbf{r}_1-\mathbf{r}_2)+\delta(\mathbf{r}_1-\mathbf{r}_2)v_{2\beta}[G^+(\mathbf{r}_2,\mathbf{r}_1)-G^-(\mathbf{r}_2,\mathbf{r}_1)]\}\ . \tag{4.3}$$

If this is integrated against any function over $\mathbf{r}_1$ or $\mathbf{r}_2$, then it is equivalent to

$$\nabla_\alpha\sigma^A_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2)=\frac{e^2}{2\pi}\delta(\mathbf{r}_1-\mathbf{r}_2)\{(\nabla_{2\beta}-\nabla_{1\beta})\mathrm{Re}[G^+(\mathbf{r}_1,\mathbf{r}_2)-G^-(\mathbf{r}_1,\mathbf{r}_2)]\}\ . \tag{4.4}$$

This need not be zero in the absence of time-reversal symmetry. This means that we can no longer integrate by parts and express the current solely in terms of the voltages at the leads. It brings into question the validity of our technique for evaluating long-range diagrams and eliminating them by expressing the current in terms of the classical electric field.

We can evaluate $\nabla_{1\alpha}\sigma^A_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2)$ using perturbation theory, and it turns out that its contribution is higher order in the disorder than the terms that we are keeping. The proof of this is similar to the proof in Ref. 10 that diagrams with no change in the sign of the energy at current vertices are higher order. $\nabla_{1\alpha}\sigma^A_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2)$ can be depicted by the diagram shown in Fig. 3(a). Fluctuations in this quantity can be investigated by calculating moments such as

$$\langle\nabla_{1\alpha}\delta\sigma^A_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2)\delta\sigma^{A,S}_{\gamma\delta}(\mathbf{r}_3,\mathbf{r}_4)\rangle$$

or

$$\langle\nabla_{1\alpha}\delta\sigma^A_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2)\nabla_{3\gamma}\delta\sigma^A_{\gamma\delta}(\mathbf{r}_3,\mathbf{r}_4)\rangle\ .$$

Some typical leading-order diagrams for these quantities are shown in Figs. 3(b) and 3(c). Their evaluation leads to

$$\begin{aligned}\langle\nabla_{1\alpha}\delta\sigma_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2)\delta\sigma_{\gamma\delta}(\mathbf{r}_3,\mathbf{r}_4)\rangle&\propto\frac{1}{E_F\tau}\left[\frac{e^2}{h}\right]^2\delta(\mathbf{r}_1-\mathbf{r}_2)\delta(\mathbf{r}_3-\mathbf{r}_4)d(\mathbf{r}_2,\mathbf{r}_4)\nabla_{2\beta}d(\mathbf{r}_2,\mathbf{r}_4)\ ,\\ \langle\nabla_{1\alpha}\delta\sigma_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2)\nabla_{3\gamma}\delta\sigma_{\gamma\delta}(\mathbf{r}_3,\mathbf{r}_4)\rangle&\propto\frac{1}{(E_F\tau)^2}\left[\frac{e^2}{h}\right]^2\delta(\mathbf{r}_1-\mathbf{r}_2)\delta(\mathbf{r}_3-\mathbf{r}_4)d(\mathbf{r}_2,\mathbf{r}_4)\nabla_{2\beta}\nabla_{4\delta}d(\mathbf{r}_2,\mathbf{r}_4)\ .\end{aligned} \tag{4.5}$$

Clearly, the fluctuations in $\nabla_{1\alpha}\sigma^A_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2)$ are smaller in an expansion in $(E_F\tau)^{-1}$ than the terms that we are keeping, so effectively we may regard $\sigma^A_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2)$ as being divergenceless in our approximation. This means that all of the techniques mentioned above to treat the symmetric part of the conductivity can be used to treat the antisymmetric part. We can calculate the long-range diagrams and eliminate them with the classical electric field. In Ref. 14 the method of Ref. 6 was extended to calculate the antisymmetric part of the voltage fluctuations in a four-lead device. As we have seen in Sec. II, this is equivalent to assuming a divergenceless conductivity tensor. Our considerations in this section support their results.

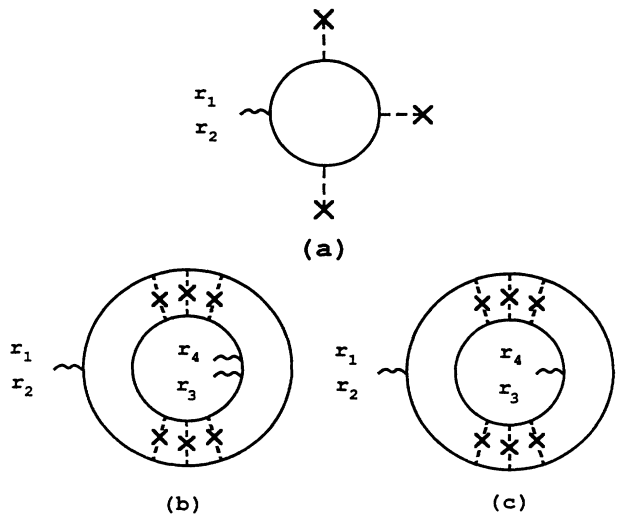

FIG. 3. (a) Diagram representing $\nabla_{1\alpha}\sigma^{A}_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2)$. (b) Diagram for $\langle \nabla_{1\alpha}\delta\sigma^{A}_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2)\delta\sigma^{A,s}_{\gamma\delta}(\mathbf{r}_3,\mathbf{r}_4)\rangle$. (c) Diagram for $\langle \nabla_{1\alpha}\delta\sigma^{A}_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2)\nabla_{3\gamma}\delta\sigma^{A}_{\gamma\sigma}(\mathbf{r}_3,\mathbf{r}_4)\rangle$.

V. RESULTS

In this section we present some results for three, four-, and six-lead devices. These calculations were done analytically by calculating the diffusion propagator in each geometry and using (2.8). Similar results have been obtained in the numerical simulations of Baranger *et al.*[18] Some details of the calculations and analytic expressions of the results are presented in Appendix C.

We first discuss the fluctuations in the transmission amplitudes in the three-probe device shown in Fig. 2 in which two of the leads are of the same length, L, and the third lead is a different length, αL. In this case we work at zero temperature, so that inelastic effects can be ignored. As mentioned above, the fluctuations in the transition amplitudes are universal in the sense that they are independent of the overall scale size, so they depend only on the ratio α and not on L. There are three independent transmission amplitudes, G_{xy}, G_{yz}, and G_{xz} (current conservation requires that $G_{xx}=-G_{xy}-G_{xz}$). In Fig. 4 we show the various correlation functions of the transmission amplitudes.

One can understand the geometrical dependence of these objects by considering the reservoirs at the ends of the leads as inelastic scattering centers where electrons lose their phase coherence. Consider, for example, $\langle \delta G^2_{xy}\rangle$. When α is very small, then lead z is very short and electrons cannot coherently travel from lead x to lead y, so fluctuations in G_{xy} go to zero. In our approximation of quasi-one-dimensional segments all electrons would escape into lead z. If there were a finite width, then some electrons would be able to pass lead z and the fluctuations would not go all of the way to zero. The fluctuations should approach a constant value when the length of the lead becomes comparable to its width. As α increases, it becomes less likely for the electrons to leave through lead z, so $\langle \delta G^2_{xy}\rangle$ increases. Similarly, when α is small, $\langle \delta G^2_{xz}\rangle=\frac{8}{15}$, which is the value of the universal conductance fluctuations in a one-dimensional two-lead conductor. This is because electrons never coherently enter lead y, so leads x and y can be thought of as independent. As α increases, it becomes more likely for electrons to enter lead y and inelastically scatter, so $\langle \delta G^2_{xz}\rangle$ decreases. Correlations between different transmission amplitudes have a finite value which is somewhat less than the size of the fluctuations because the electrons are exploring different regions of the sample.

We have also calculated the dependence of voltage fluctuations at $T=0$ on the length of the voltage probes. Using the same three-probe geometry as above, we find that the voltage fluctuation $\langle \delta V^2_{xz,xy}\rangle$ is essentially linear in α for $\alpha>1$. That is, the voltage fluctuations are linear in the length of the voltage probe, provided the length is shorter than the inelastic length. The fact that the fluctuations increases with α is no surprise if one thinks of the fluctuations as coming from electric fields in the voltage probe; however, one might expect the dependence to be quadratic rather than linear. The dependence on the

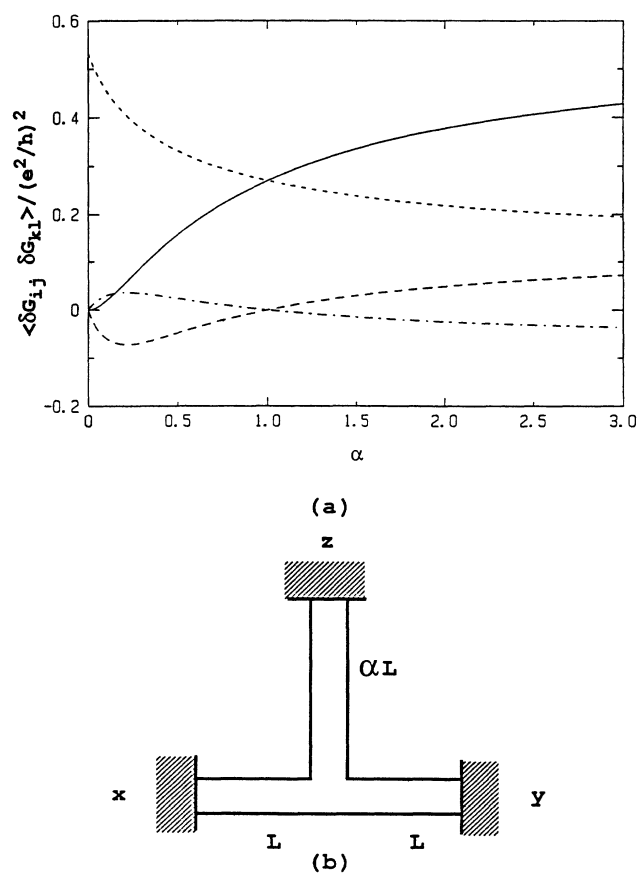

FIG. 4. (a) Fluctuations in the transmission amplitudes expressed in units of e^2/h for the three-lead device shown in (b). $\langle \delta G^2_{xy}\rangle$, solid line; $\langle \delta G^2_{xz}\rangle=\langle \delta G^2_{yz}\rangle$, short-dashed line; $\langle \delta G_{xz}\delta G_{yz}\rangle$, long-dashed line; $\langle \delta G_{xz}\delta G_{xy}\rangle=\langle \delta G_{yz}\delta G_{xy}\rangle$, long-dashed–short-dashed line.

length of the voltage leads at low temperature is similar in devices with more leads. It increases linearly until the length of the lead reaches the inelastic length, at which point the effect saturates and the voltage fluctuations become independent of the length of the voltage leads.

We now present some results for four- and six-lead devices in the limit that the lengths of the current and voltage leads are much longer than the inelastic length. This situation is more relevant to the recent experiments.[4,5] The samples are specified by the ratios of the lengths of the segments between the leads to the inelastic length. The results for the voltage fluctuation as a function of the distance between the voltage probes are shown in Fig. 5. The solid line represents a four-probe device in which the distance between voltage probes is varied. The individual points are from a fixed six-probe device in which the voltage is measured between different combinations of leads. There are two sets of data, which correspond to the same sample at two different temperatures.

The voltage fluctuations approach a finite value as the distance between the voltage leads goes to zero. This can be understood in terms of the fluctuating electric fields in the leads. As the distance between the voltage leads is increased, the fluctuations increase linearly. In this sense, the fluctuations are not length independent since there is a finite slope. This is consistent with the data shown in Refs. 4 and 5. The fluctuations shown in Fig. 5 are for zero magnetic field. To compare with measurements of the symmetric part of the fluctuations in a magnetic field, one should divide by 2 because of the elimination of the particle-particle channel. This also agrees well with experiment. The result for the four-probe case is the same as that in Ref. 1, except that finite-length voltage probes were included there, which effectively increases the amount of inelastic scattering, since electrons can escape out of the leads. One should expect a finite slope because, as the leads are moved apart, there are also fluctuations that arise due to the electric fields in the segment between the leads. The antisymmetric part of the fluctuations have been calculated in Ref. 14. There it was shown that the segment between the leads does not contribute to the antisymmetric fluctuations. Therefore, there is a much weaker dependence on the separation between the voltage leads, and the fluctuations remain roughly constant even when the leads are further apart than L_{in}.

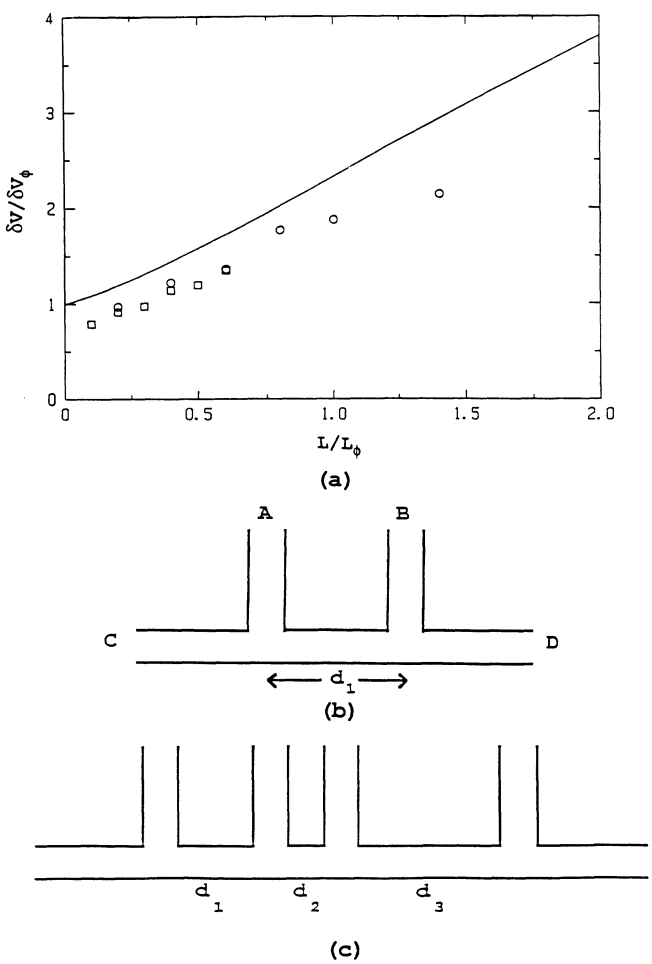

FIG. 5. (a) Voltage fluctuations as a function of distance between voltage leads. The solid line is from the four-lead device shown in (b). The individual points are from the six-lead device shown in (c). Circles, $(d_1,d_2,d_3)/L_{in}=(0.4,0.2,0.8)$; squares, $(d_1,d_2,d_3)/L_{in}=(0.2,0.1,0.4)$.

The only difference between the four-lead case and the six-lead case is that for the six-lead case there are unused voltage leads. The effect of these is to increase the amount of inelastic scattering because electrons can diffuse into these leads, so it will take them longer to go from one place to another. Therefore, we expect the voltage fluctuations to decrease when there are extra leads. This effect will be greater when the leads have length comparable to the inelastic length, since in that case electrons will be more likely to escape through the unused leads. At this point there will be a crossover to the linear behavior described above.

VI. CONCLUSION

Voltage leads play a very important role in determining the fluctuations of voltage measured in multilead devices. This is a result of the highly nonlocal nature of the correlations responsible for these fluctuations. There are two equivalent ways of understanding these results. The fluctuating-electric-field approach[6] emphasizes the fact that there can be electric fields in the voltage leads where there is no current and provides a direct means of obtaining Eq. (2.8). The classical electric field must be used in these calculations in order to avoid the classical long-range behavior of the conductivity. The fluctuating-transmission-amplitude approach[8] is a general method for treating arbitrary multilead geometries and shows the relation between multilead voltage fluctuations and the universal conductance fluctuations found in two-lead devices.

In the presence of a magnetic field, these calculations apply to the symmetric part of the conductivity. The antisymmetric part of the conductivity is not divergenceless, so that before averaging one cannot treat the problem in terms of transmission amplitudes. When evalu-

ated in perturbation theory, however, the effect of this is higher order in the disorder, so that the techniques presented here can be used.

The fluctuations in the transmission amplitudes show geometrical dependence as a result of the role of the resevoirs at the ends of the leads as inelastic scattering centers. Voltage fluctuations also reflect this effect by being smaller in the presence of extra leads. Voltage fluctuations approach a constant value as the distance between the voltage leads goes to zero, however, they do not appear to be length independent for distances less than the inelastic length, since there is a finite slope in the fluctuations as a function of distance between the voltage leads.

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APPENDIX A

In this appendix we show how Eq. (3.9) follows from (3.8). That is, we show how in a sample composed of quasi-one-dimensional segments the internal segments can be eliminated in the sum in (3.8) in favor of changing the region of integration. We demonstrate this for the four-lead device in Fig. 1; however, it is straightforward to generalize to any number of leads, provided the sample is simply connected in the sense that there is a single well-defined path connecting any two points in the sample (i.e., no loops).

Our starting point is Eq. (3.8),

$$\langle \delta G_{ij}\delta G_{kl}\rangle = \sum_{i',j',k',l'} \phi_{ii'}\phi_{jj'}\phi_{kk'}\phi_{ll'} \int d\mathbf{r}_{i'} \int d\mathbf{r}_{j'} \int d\mathbf{r}_{k'} \int d\mathbf{r}_{l'} \Gamma(\mathbf{r}_{i'},\mathbf{r}_{j'}\mathbf{r}_{k'}\mathbf{r}_{l'}) \ . \tag{A1}$$

The $i', \ldots, l'$ can refer to the external leads, a,b,c,d, or the internal segment, which we will refer to as m. By current conservation we can express

$$\phi_{im} = \phi_{ia} + \phi_{ic} \tag{A2}$$

for any external lead i. Considering only the i' sum, we can then write

$$\begin{aligned}\langle \delta G_{ij}\delta G_{kl}\rangle &= \sum_{j',k',l'} \phi_{jj'}\phi_{kk'}\phi_{ll'} \int d\mathbf{r}_{j'} \int d\mathbf{r}_{k'} \int d\mathbf{r}_{l'} \left[\sum_{i'=a,b,c,d} \phi_{ii'} \int d\mathbf{r}_{i'} \Gamma(\mathbf{r}_{i'},\mathbf{r}_{j'},\mathbf{r}_{k'},\mathbf{r}_{l'}) + \sum_{i'=a,c} \phi_{ii'} \int d\mathbf{r}_m \Gamma(\mathbf{r}_m,\mathbf{r}_{j'},\mathbf{r}_{k'},\mathbf{r}_{l'}) \right] \\ &= \sum_{j',k',l'} \phi_{jj'}\phi_{kk'}\phi_{ll'} \int d\mathbf{r}_{j'} \int d\mathbf{r}_{k'} \int d\mathbf{r}_{l'} \sum_{i'=a,b,c,d} \phi_{ii'} \int_x^{i'} d\mathbf{r}_1 \Gamma(\mathbf{r}_1,\mathbf{r}_{j'},\mathbf{r}_{k'}\mathbf{r}_{l'}) \ ,\end{aligned} \tag{A3}$$

where x is the point where the leads b and d are attached to the internal segment. Actually, the result is independent of x, since moving x only changes the integral by a constant, and current conservation requires that $\sum_{i'}\phi_{ii'}=0$. This procedure can be repeated on the other variables and we obtain

$$\langle \delta G_{ij}\delta G_{kl}\rangle = \frac{1}{\sigma_0^4} \sum_{i',j',k',l'} \langle G_{ii'}\rangle \langle G_{jj'}\rangle \langle G_{kk'}\rangle \langle G_{ll'}\rangle \int_x^{i'} d\mathbf{r}_1 \int_x^{j'} d\mathbf{r}_2 \int_x^{k'} d\mathbf{r}_3 \int_x^{l'} d\mathbf{r}_4 \Gamma(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4) \ . \tag{A4}$$

APPENDIX B

In this appendix we show explicitly how to extract the transmission amplitudes from voltage measurements on a three-lead device, whose leads are labeled x, y, and z as in Fig. 4(b). There are three independent voltage measurements which can be made: $V_{xy,xz}$, $V_{xy,yz}$, and $V_{xz,yz}$. These voltages are determined by the relation

$$I_i = \sum_j G_{ij} V_j \ . \tag{B1}$$

$V_{xy,yz}$ and $V_{xz,yz}$ can be determined from the equations

$$\begin{aligned} I_x &= 0 = G_{xy} V_{xy,yz} + G_{xz} V_{xz,yz} \ , \\ I_y &= I = -(G_{xy}+G_{yz}) V_{xy,yz} + G_{yz} V_{xz,yz} \ , \end{aligned} \tag{B2}$$

where we have chosen the zero potential in (B1) to be lead x and used the fact that $G_{yy} = -G_{xy} - G_{yz}$. The equation for $Iz = -I$ is a linear combination of these equations. These can be solved for $V_{xy,yz}$ and $V_{xz,yz}$:

$$\begin{aligned} V_{xy,yz} &= \frac{G_{xz}}{G_{xy}G_{yz} + G_{xz}G_{yz} + G_{xy}G_{xz}} I \ , \\ V_{xz,yz} &= \frac{G_{xy}}{G_{xy}G_{yz} + G_{xz}G_{yz} + G_{xy}G_{xz}} I \ . \end{aligned} \tag{B3}$$

Similarly, it can be shown that

$$V_{xy,xz} = \frac{G_{yz}}{G_{xy}G_{yz} + G_{xz}G_{yz} + G_{xy}G_{xz}} I \ .$$

These can easily be inverted to solve for G_{ij}, and one finds

$$G_{xy}=\frac{V_{xz,yz}}{V_{xz,yz}V_{xy,xz}+V_{xz,yz}V_{xy,yz}+V_{xy,xz}V_{xy,yz}}I\ ,$$
$$G_{yz}=\frac{V_{xy,xz}}{V_{xz,yz}V_{xy,xz}+V_{xz,yz}V_{xy,yz}+V_{xy,xz}V_{xy,yz}}I\ , \tag{B4}$$
$$G_{xz}=\frac{V_{xy,yz}}{V_{xz,yz}V_{xy,xz}+V_{xz,yz}V_{xy,yz}+V_{xy,xz}V_{xy,yz}}I\ .$$

APPENDIX C

In this appendix we present some details of the calculations of the results presented in Sec. III. All results depend only on the diffusion propagator $d(\mathbf{r},\mathbf{r}')$ which must be calculated in each geometry and satisfies

$$(-\nabla^2+L_{\rm in}^{-2})d(\mathbf{r},\mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}')\ , \tag{C1}$$

subject to the appropriate boundary conditions. In samples composed of quasi-one-dimensional segments, $d(\mathbf{r},\mathbf{r}')$ will be constant across the cross sections, so it can be found by solving a one-dimensional diffusion equation in each segment,

$$\left[-\frac{\partial^2}{\partial x_i^2}+L_{\rm in}^{-2}\right]d(x_i,x_j')=\delta(x_i-x_j') \tag{C2}$$

(x_i is the position along segment i) subject to the boundary condition that d vanish at the end of a lead, and to the matching condition that at very junction (1) d should be continuous, and (2) the sum of the derivatives of d is zero (so that no net current flows into the junction).

We shall now describe the solutions for the three cases presented in Sec. III.

1. Three-lead device at $T=0$

By solving (C2) in the three-lead geometry shown in Fig. 4(b), we obtain

$$d(x_i,x_j')=\frac{L_k}{M}x_ix_j'\quad\text{for } i\neq j\ ,$$
$$d(x_i^<,x_i^>)=x_i^<\left[1-\frac{L_j+L_k}{M}x_i^>\right]\ , \tag{C3}$$

where $M=L_xL_y+L_xL_z+L_yL_z$, and (i,j,k) is any permutation of (x,y,z). $x_i^<$ is the closest to the end of the lead. To calculate fluctuations in the transmission amplitudes, we evaluate Eq. (3.8). From the expression for Γ, (2.7), we can write [in units of $(e^2/h)^2$]

$$\langle\delta G_{ij}\delta G_{kl}\rangle=\sum_{i',j',k',l'}\phi_{ii'}\phi_{jj'}\phi_{kk'}\phi_{ll'}(\delta_{i'j'}\delta_{k'l'}\chi_{i'k'}+\delta_{i'k'}\delta_{j'l'}\chi_{i'j'}+\delta_{i'l'}\delta_{j'k'}\chi_{i'j'})\ , \tag{C4}$$

where

$$\chi_{ij}=\int dx_i\int dx_j' d(x_i,x_j')^2\ .$$

After some lengthy algebra, we find that for the case $L_x=L_y=L$ and $L_z=\alpha L$,

$$\langle\delta G_{xy}^2\rangle=\frac{8}{45}\frac{\alpha^2(51\alpha^2+56\alpha+16)}{(1+2\alpha)^4}\ ,$$
$$\langle\delta G_{xz}^2\rangle=\langle\delta G_{yz}^2\rangle=\frac{8}{45}\frac{12\alpha^4+44\alpha^3+46\alpha^2+18\alpha+3}{(1+2\alpha)^4}\ ,$$
$$\langle\delta G_{xz}\delta G_{yz}\rangle=\frac{16}{45}\frac{\alpha(\alpha-1)(6\alpha^2+8\alpha+3)}{(1+2\alpha)^4}\ , \tag{C5}$$
$$\langle\delta G_{xy}\delta G_{xz}\rangle=\langle\delta G_{xy}\delta G_{yz}\rangle$$
$$=-\frac{8}{45}\frac{\alpha(\alpha-1)(6\alpha^2+8\alpha+3)}{(1+2\alpha)^4}\ .$$

These results are plotted in Fig. 4. Note that different transmission amplitudes have finite correlations. This contradicts Büttiker's assumption that different transmission amplitudes are uncorrelated[8]

We have calculated the dependence of the voltage fluctuations on the length of the voltage lead,

$$\langle\delta V_{xz,xy}^2\rangle=\frac{I^2}{(\sigma_0A/L)^4}\frac{8}{45}\frac{20\alpha^3+76\alpha^2+24\alpha+3}{(1+2\alpha)^2}\ . \tag{C6}$$

This is roughly linear for $\alpha>1$.

2. Four-lead device with long leads at finite temperature

The boundary conditions are simplified in this situation since there will be exponential damping of d in each lead, so that it is not necessary to account for the boundary conditions at the ends of the leads. [We do not need to know $d(x_i,x_{i'})$ where x_i and $x_{i'}$ are in the same lead].

For the geometry shown in Fig. 5(b) with $x_a,\ldots,x_d$ measured from the functions and x_m measured from the junction with leads a and b, we find (expressing lengths in units of $L_{\rm in}$)

$$d(x_a,x_b')=-\frac{2e^{d_1}}{1-9e^{2d_1}}e^{-x_a-x_b'}\ ,$$
$$d(x_a,x_c')=\frac{1-3e^{2d_1}}{1-9e^{2d_1}}e^{-x_a-x_c'}\ ,$$
$$d(x_a,x_m')=\frac{e^{-x_a}}{1-9e^{2d_1}}(e^{x_m'}-3e^{2d_1-x_m'})\ , \tag{C7}$$
$$d(x_m^<,x_m^>)=-\frac{e^{d_1}}{2(1-9e^{2d})}(e^{-x_m^<}-3e^{x_m^<})$$
$$\times(e^{-(d_1-x_m^>)}-3e^{(d_1-x_m^>)})\ .$$

Other combinations will be related to the above ones by symmetry.

We can now integrate these and use (2.8) to get

$$\langle \delta V_{ab,cd}^2 \rangle = \delta V_\phi^2 \frac{4(-8-3d_1+240d_1e^{2d_1}+243d_1e^{4d_1}+108d_1^2e^{2d_1}+240e^{2d_1}-216e^{4d_1})}{(1-9e^{2d_1})^2} , \tag{C8}$$

where

$$\delta V_\phi \equiv \frac{e^2}{h} \frac{I}{(\sigma_0 A/L_{\rm in})^2} .$$

3. Six-lead device with long leads at finite temperature

In order to calculate the diffusion propagator in a six-lead device shown in Fig. 5(c), we note that if x_i is in an external lead and is measured from the junction (again expressing lengths in units of $L_{\rm in}$)

$$d(x_i,x_j') = d(x_i=0, x_j')e^{-x_i} . \tag{C9}$$

Therefore, once $d(x,x')$ is known for x,x' in the internal segments, it is also known in the external leads. To calculate $d(x,x')$ in the internal "spine" of our sample, we write the general solution as

$$d(x_i^L,x_j^R) = a_{ij}(e^{-x_i^L}+r_ie^{x_i^L})(e^{-(d_i-x_i^R)}+s_ie^{(d_i-x_i^R)}) , \tag{C10}$$

where the superscripts L and R refer to whichever variable is further to the left or right in the "spine."' r_i and s_i can be found by exploiting the matching conditions at each junction. We find

$$r_1=-3, \quad r_{i+1}=\frac{1+3r_ie^{2d_i}}{1-r_ie^{2d_i}} ,$$
$$s_1=-3, \quad s_{i-1}=\frac{1+3s_ie^{2d_i}}{1-s_ie^{2d_i}} . \tag{C11}$$

The normalization a_{ij} is determined by first calculating the diagonal components, which can be found by demanding that the discontinuity in the slope of $d(x_i,x_i')$ be unity as x_i passes through x_i'. We find

$$a_{ii} = \frac{e^{d_i}}{2(r_is_ie^{2d_i}-1)} . \tag{C12}$$

The matching conditions at each junction can then be used to determine the remaining a_{ij} from

$$a_{ij} = \frac{2e^{d_j}}{1-s_je^{2d_j}} a_{ij-1}$$
$$= \frac{2e^{d_i}}{1-r_ie^{2d_i}} a_{i-1j} . \tag{C13}$$

The coefficients a_{ij}, r_i, s_j can now be calculated. Once the coefficients have been calculated, the entire diffusion propagator is known, and the voltage fluctuations can be calculated. We carried it out for six external leads, though the procedure is valid for any number of leads. The algebra was carried out by computer and the results are very complicated, so they will be omitted. We evaluated the expression for $(d_1,d_2,d_3)/L_{\rm in}=(0.4,0.2,0.8)$ and (0.2, 0.1, 0.4), which could correspond to the same sample at two different temperatures. The results are shown in Fig. 5.

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