# Voltage fluctuations in mesoscopic metal rings and wires

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We apply a recently developed multiterminal formalism to compute the fluctuations in the transport properties of mesoscopic rings. The voltage fluctuations $\delta V$ are not symmetric with respect to reversal of the magnetic field **H**. Contributions to the part of $\delta V$ symmetric with respect to reversal of **H** come from the whole length of the sample, while the antisymmetric part of $\delta V$ comes only from the region of the junction between the voltage and current leads. We show that the Aharonov-Bohm (AB) oscillation decreases exponentially with $C_{\rm ring}/L_{\rm in}$ ($C_{\rm ring}$ denotes the circumference of the ring, $L_{\rm in}$ denotes the inelastic mean free path). The AB oscillations in the voltage are not symmetric with respect to zero **H** field—in the highly coherent regime ($C_{\rm ring}/L_{\rm in}<1$) the phase of this oscillation can take on any value, while in the classical limit the phase becomes symmetric—0 or $\pi$. We demonstrate the existence of nonlocal voltage fluctuations, ones in which the current path and voltage path do not intersect; both aperiodic fluctuations and AB oscillations fall off exponentially with $d/L_{\rm in}$, where $d$ is the closest approach of the current and voltage paths. Finally, we obtain new results on the energy correlation $E_c$ for the AB effect, finding it to be larger than previously expected.

## I. INTRODUCTION

Last year, experimental results cast doubt on the universal-conductance-fluctuation theory[1–3] which asserts that for any mesoscopic device (one for which the inelastic scattering length $L_{\rm in}$ is comparable to the device size), the sample-to-sample fluctuations in the conductance are a constant of order unity independent of the conductance or the overall size of the device. Benoit *et al.*[4] found conductance fluctuations which, far from being of order unity, could be as large as one million. It was clear[5] that the difficulty lay in the fact that the measurement involved separate current and voltage probes, while the theory assumed that these probes were the same. A recent series of papers,[6–15] marking a significant advance in the theory of mesoscopic conductors, have properly accounted for the four-terminal nature of the experiments.

It is the purpose of the present paper to apply this new formalism to a variety of different devices, including metal rings and wires. For the first time, we can make predictions for these devices which are directly comparable to experiment. For example, the new four-terminal formalism is capable of predicting the magnetic field asymmetry of the measured four-terminal resistances, unlike the old two-terminal theory in which only the symmetric response was accounted for.

Here we summarize the major results of the present calculation: In the following, $\delta V_{ij,kl}$ refers (see Fig. 1) to the fluctuations in the voltage measured between leads $i$ and $j$ when a current $I$ is introduced at lead $k$ and taken out at lead $l$.

(1) We reconfirm for metal rings [Fig. 1(a)] as well as wires [Fig. 1(b)] that the aperiodic part of $\delta V_{14,23}$ symmetric with respect to reversal of the magnetic field grows like $(C_{\rm ring}/L_{\rm in})^{1/2}$ for large $C_{\rm ring}/L_{\rm in}$, and the antisymmetric part is length independent.[4,8,11–13,16] Here $C_{\rm ring}$ is the circumference of the ring.

(2) We show that different voltage fluctuations measured by swapping a pair of current and voltage leads, i.e., $\langle \delta V_{14,23}\delta V_{13,24}\rangle$, are strongly correlated in the symmetric part [especially when $(C_{\rm ring}/L_{\rm in})^{1/2}$ is large], but are uncorrelated in the antisymmetric part. This is supported by experimental observations.[17]

(3) Aharonov-Bohm (AB) oscillations[18–24] in $\delta V_{13,24}$ asymptotically approach an exponential decay $\exp(-\alpha C_{\rm ring}/L_{\rm in})$ as $C_{\rm ring}/L_{\rm in}$ gets large. However, $C_{\rm ring}/L_{\rm in}$ must be on the order of 3 before this asymptotic result applies. The symmetric part decays a little more slowly (i.e., smaller $\alpha$) than the antisymmetric part.[5]

(4) We have computed the probability distribution of the phase of the AB oscillation around zero magnetic field. This phase is not exactly zero or $\pm\pi$ because

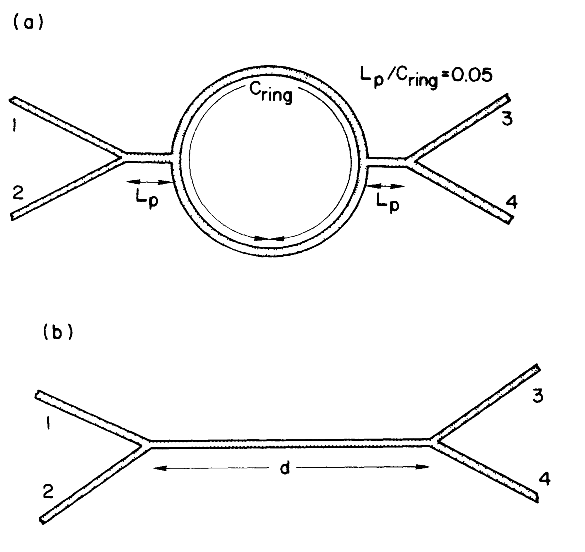



FIG. 1. (a) The geometry of the ring which we study in our calculations. $L_p$ is the distance from the ring to the voltage probes, and we take it to be about 0.05 of the circumference of the ring, $C_{ring}$. This is the experimental geometry used by Benoit *et al.* (Ref. 17). With our approximations, all the wires are quasi-one-dimensional, and the angle between the voltage and current wires is irrelevant. (b) The geometry of the wire studied in our calculation.

$\delta V_{13,24}$ has an antisymmetric part.[5,17] For small $C_{ring}/L_{in}$, the phase is equally likely to be any angle between $-\pi$ and $\pi$. As $C_{ring}/L_{in}$ grows, the phase slowly develops a preference for angles near 0 and $\pm\pi$.

(5) We have computed the "nonlocal"[14,15,17,25] voltage fluctuation $\delta V_{12,34}$ for both a ring and a wire (Fig. 1). In

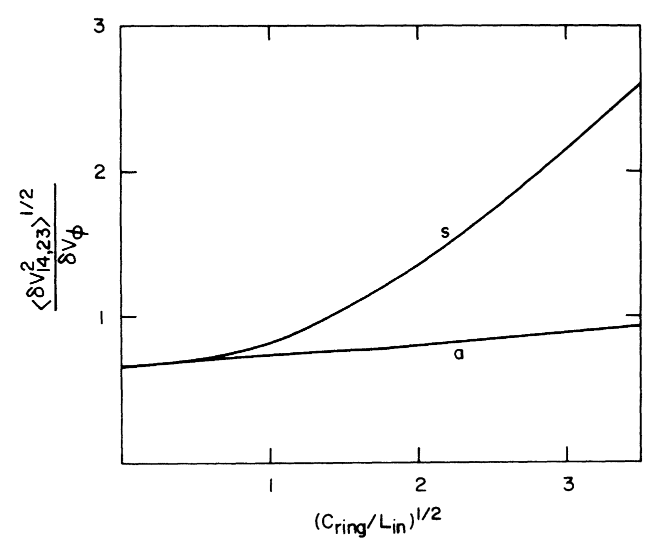



FIG. 2. (a) The symmetric ($s$) and antisymmetric ($a$) rms voltage fluctuations $\langle \delta V^2_{14,23} \rangle / \delta V_\phi$ [$\delta V_\phi$ is defined in Eq. (12)], vs $(C_{ring}/L_{in})^{1/2}$. As noted in previous publications, (Refs. 8, 12, and 13) the symmetric part rises like $(C_{ring}/L_{in})$, while the antisymmetric part asymptotes to a constant.

both cases these decay exponentially with the distance between the current path and the voltage path (defined as the shortest path through the device from one voltage probe to the other). For the case of the wire we find $\delta V_{12,34} \sim \exp(-\alpha d/L_{in})$, where $\alpha \approx 1.1$. (The exact asymptotic result is $\alpha = 1$.) Experiment[26] finds a similar exponential decay. In a ring this decay is approximately $\exp(-0.6C_{ring}/L_{in})$. The Aharonov-Bohm oscillations also decay away with the same decay constant. The symmetric and antisymmetric channels are identical for this nonlocal measurement, which means that the Aharonov-Bohm phase mentioned in (3) above is always equally likely to be any angle between $-\pi$ and $\pi$.

(6) We have studied the energy correlation function,[1,15] and have extracted the correlation length $E_c$, for voltage fluctuations of a ring. Physical arguments which relate $E_c$ to the time required for a carrier to diffuse one inelastic mean free path suggest[1] that $E_c \simeq \hbar D/L_{in}^2$ for the aperiodic voltage fluctuations. We confirm this result in the present calculation. Similar physical arguments advanced by Milliken *et al.*[27] say that the $E_c$ appropriate for the Aharnov-Bohm oscillations should be related to the time required for the carrier to diffuse around the entire ring, not just one inelastic scattering length, so that $E_c \simeq \hbar D/C_{ring}^2$, where $C_{ring}$ is the circumference of the ring. Surprisingly, our calculations do *not* confirm this reasoning; we can force our results for $E_c$ to fit a functional form $E_c \simeq \hbar D/L_{in}^{\alpha} C_{ring}^{2-\alpha}$, but we find $\alpha \approx 1.3$. We will present below a more refined physical argument which would suggest that this new functional form is correct, and which predicts roughly $\alpha = 1$.

## II. FORMALISM

Most of the basic formalism has already been presented in our earlier paper[8] but we will review it here and introduce some new results relating to the symmetric and antisymmetric response.

The basic statement of linear response theory is

$$J_\alpha(\mathbf{r}) = \int d\mathbf{r}' \, \sigma_{\alpha\beta}(\mathbf{r},\mathbf{r}') E_\beta(\mathbf{r}') \ . \tag{1}$$

Here $J$ is the current density, $E$ is a local electric field, and $\sigma$ is the nonlocal conductivity which for the case of finite magnetic field is given by the Streda formula.[28] As Kane *et al.*[8] showed, by using the result $\partial_\alpha \sigma_{\alpha\beta} = 0$ [which is true to lowest order in $(k_F = l)^{-1}$, with $k_F$ denoting the Fermi wave vector and 1 denoting the elastic mean free path], introducing a potential $\partial_\beta V = E_\beta$, and integrating by parts, we can turn this into the basic equation introduced by Büttiker:[5,9]

$$I_i = G_{ij} V_j \ , \tag{2}$$

where $I_j$ is the current flowing in through lead $i$ (see Fig. 1), $V_j$ is the voltage at lead $j$, and

$$G_{ij} = \int dS_{i\alpha} \int dS'_{j\beta} \, \sigma_{\alpha\beta}(\mathbf{r},\mathbf{r}') \ . \tag{3}$$

In Eq. (3), the integral over $dS_{i\alpha}$ is over the surface along which lead $i$ joins to the device. We will be interested in the fluctuations in the measured four-terminal voltages $\delta V_{ij,kl}$. As shown by Kane *et al.*,[8] these are related to fluctuations in the conductances $\delta G_{ij}$ by a suitable inversion:

$$\delta V_{ij,kl} = -I \sum_{m,n} (\langle R_{im}\rangle - \langle R_{jm}\rangle) \times \delta G_{mn}(\langle R_{nk}\rangle - \langle R_{nl}\rangle) . \quad (4)$$

Here $\langle R_{ij}\rangle$ is the matrix inverse of $\langle G_{ij}\rangle$, the ensemble-averaged conductance tensor $G_{ij}$. Actually the matrix $G_{ij}$ is singular,[8,29] which means that only the differences $\langle R_{im}\rangle - \langle R_{jm}\rangle$ which appear in Eq. (4) are well defined.

Diagrammatic analysis permits us to calculate fluctuations in the nonlocal conductivity, and thus in the measured four-terminal resistances. This has been discussed in Kane *et al.*;[8] the results (including some new results on the antisymmetric fluctuations) are

$$\langle \delta\sigma^{s,a}_{\alpha\beta}(\mathbf{r}_1,\mathbf{r}_2,E_F,\mathbf{A})\delta\sigma^{s,a}_{\gamma\delta}(\mathbf{r}_3,\mathbf{r}_4,E_F+\Delta E_F,\mathbf{A}+\Delta\mathbf{A})\rangle = \int d\mathbf{r}'_1 \int d\mathbf{r}'_2 \int d\mathbf{r}'_3 \int d\mathbf{r}'_4 \phi_{\alpha\alpha'}(\mathbf{r}_1,\mathbf{r}'_1)\phi_{\beta\beta'}(\mathbf{r}_2,\mathbf{r}'_2)\phi_{\gamma\gamma'}(\mathbf{r}_3,\mathbf{r}'_3)\phi_{\delta\delta'}(\mathbf{r}_4,\mathbf{r}'_4) \times \Gamma^{(s,a),(s,a)}_{\alpha'\beta'\gamma'\delta'}(\mathbf{r}'_1,\mathbf{r}'_2,\mathbf{r}'_3,\mathbf{r}'_4,E_F,E_F+\Delta E_F,\mathbf{A},\mathbf{A}+\Delta\mathbf{A}) . \quad (5)$$

We have accounted for the possibility that we are correlating the conductivities measured at different Fermi energies $E_F$ and $E_F+\Delta E_F$ or in the presence of different vector potentials $\mathbf{A}$ and $\mathbf{A}+\Delta\mathbf{A}$. Here $\phi_{\alpha\beta}(\mathbf{r},\mathbf{r}')\equiv\delta_{\alpha\beta}\delta(\mathbf{r}-\mathbf{r}')-\nabla_\alpha\nabla'_\beta d(\mathbf{r},\mathbf{r}')$ is the "flow function" introduced in Kane *et al.*, and [8] $d(\mathbf{r},\mathbf{r}')$ is a diffusion propagator satisfying the equation $-\nabla^2 d(\mathbf{r},\mathbf{r}')=\delta(\mathbf{r}-\mathbf{r}')$. The boundary conditions are: $d(\mathbf{r},\mathbf{r}')=0$ when $\mathbf{r}$ and $\mathbf{r}'$ are at the surfaces where the device joins onto perfect leads, and $\hat{n}\cdot\nabla d(\mathbf{r},\mathbf{r}')=0$ on the insulating boundary of the sample. $\Gamma^{ss}_{\alpha\beta\gamma\delta}$ and $\Gamma^{aa}_{\alpha\beta\gamma\delta}$ are a sum of diagrams of the form discussed in Kane *et al.*[8] The result is

$$\Gamma^{ss}_{\alpha\beta\gamma\delta}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4,\Delta E_F,\mathbf{A},\mathbf{A}+\Delta\mathbf{A}) = 8\left[\frac{e^2}{h}\right]^2 \delta_{\alpha\beta}\delta_{\gamma\delta}\delta(\mathbf{r}_1-\mathbf{r}_2)\delta(\mathbf{r}_3-\mathbf{r}_4)[\,|\tilde{d}(\mathbf{r}_1,\mathbf{r}_3,\Delta E_F,\Delta\mathbf{A})|^2+|\tilde{d}(\mathbf{r}_1,\mathbf{r}_3,\Delta E_F,2\mathbf{A}+\Delta\mathbf{A})|^2]$$
$$+\delta_{\alpha\gamma}\delta_{\beta\delta}\delta(\mathbf{r}_1-\mathbf{r}_3)\delta(\mathbf{r}_2-\mathbf{r}_4)[\,|\tilde{d}(\mathbf{r}_1,\mathbf{r}_2,\Delta E_F,\Delta\mathbf{A})|^2+|\tilde{d}(\mathbf{r}_1,\mathbf{r}_2,\Delta E_F,2\mathbf{A}+\Delta\mathbf{A})|^2]$$
$$+\delta_{\alpha\delta}\delta_{\beta\gamma}\delta(\mathbf{r}_1-\mathbf{r}_4)\delta(\mathbf{r}_2-\mathbf{r}_3)[\,|\tilde{d}(\mathbf{r}_1,\mathbf{r}_2,\Delta E_F,\Delta\mathbf{A})|^2+|\tilde{d}(\mathbf{r}_1,\mathbf{r}_2,\Delta E_F,2\mathbf{A}+\Delta\mathbf{A})|^2] , \quad (6)$$

$$\Gamma^{aa}_{\alpha\beta\gamma\delta}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4,\delta E_F,\mathbf{A},\mathbf{A}+\Delta\mathbf{A}) = 8\left[\frac{e^2}{h}\right]^2 [\delta_{\alpha\gamma}\delta_{\beta\delta}\delta(\mathbf{r}_1-\mathbf{r}_3)\delta(\mathbf{r}_2-\mathbf{r}_4)-\delta_{\beta\gamma}\delta_{\alpha\delta}\delta(\mathbf{r}_2-\mathbf{r}_3)\delta(\mathbf{r}_1-\mathbf{r}_4)] \times[\,|\tilde{d}(\mathbf{r}_1,\mathbf{r}_2,\Delta E_F,\Delta\mathbf{A})|^2-|\tilde{d}(\mathbf{r}_1,\mathbf{r}_2,\Delta E_F,2\mathbf{A}+\Delta\mathbf{A})|^2] . \quad (7)$$

The diffusion propagator $\tilde{d}$ in Eqs. (6) and (7) is cut off by the inelastic scattering length, the Fermi energy difference, and the vector potential, and so satisfies the equation

$$[(-i\nabla+e\mathbf{A})^2+L_{\rm tot}^{-2}]\tilde{d}(\mathbf{r},\mathbf{r}',\Delta E_F,\mathbf{A})=\delta(\mathbf{r}-\mathbf{r}') . \quad (8)$$

The boundary conditions are the same as for $d(\mathbf{r},\mathbf{r}')$. Here

$$L_{\rm tot}^{-2}\equiv iL_E^{-2}+L_{\rm in}^{-2},\quad L_E^{-2}\equiv\frac{\Delta E_F}{\hbar D} , \quad (9)$$

$L_{\rm in}$ is the inelastic scattering length, and $D$ is the diffusion coefficient. The diffusion propagator with argument $2\mathbf{A}+\Delta\mathbf{A}$ corresponds to the particle-particle contribution to the fluctuations, while the diffusion propagator with argument $\Delta\mathbf{A}$ corresponds to the particle-hole contribution. We will ignore the particle-particle pieces when the magnetic flux is large, i.e., when $\mathbf{A}$ is large.

The diagrammatic analysis shows that there is some degree of universality in the quantities discussed above. Putting together Eqs. (3) and (5)–(7), the first statement of universality[9] is for the conductance tensor fluctuation:

$$\langle \delta G_{ij}(L_{\rm in},E_F,\mathbf{A})\delta G_{kl}(L_{\rm in},E_F+\Delta E_F,\mathbf{A}+\Delta\mathbf{A})\rangle = \left[\frac{e^2}{h}\right]^2 r_{ijkl}(L/L_{\rm tot},\mathbf{A},\mathbf{A}+\Delta\mathbf{A}) . \quad (10)$$

This says that the conductance fluctuation depends on a (dimensionless, and therefore of order unity) function $r$. $r(\ )$ is a function of the lead indices $i$, $j$, $k$, and $l$, of the total scattering length $L_{\rm tot}$ (defined above) relative to the sample dimension $L$, of the vector potentials $\mathbf{A}$ and $\mathbf{A}+\Delta\mathbf{A}$, and of the sample shape. The explicit form of the function $r(\ )$ can be read off from Eqs. (6) and (7).

What makes it universal is what it does *not* depend on: it depends on the inelastic scattering length and the Fermi energy difference only through the single length $L_{tot}$ defined above; all that matters is the size of this length relative to the sample dimensions. This function is also independent of the average conductivity of the sample. It is precisely this independence of overall sample size and average conductivity which embodied the previous concept of universality,[1] which is now stated here in a more elaborate form.

We can also write a "universal" equation for the voltage fluctuation:

$$\langle \delta V_{ab,cd}(L_{in},E_F,\mathbf{A})\delta V_{ef,gh}(L_{in},E_F+\Delta E_F,\mathbf{A}+\Delta\mathbf{A})\rangle = \delta V_\phi^2 s_{ab,cd;ef,gh}(L/L_{tot},\mathbf{A},\mathbf{A}+\Delta\mathbf{A}) . \quad (11)$$

Here

$$\delta V_\phi \equiv \frac{e^2}{h}\frac{I}{G_\phi^2} \equiv \frac{e^2}{h}\frac{I}{\left[\dfrac{\sigma_0\mathcal{A}}{|L_{tot}|}\right]^2} , \quad (12)$$

where $\sigma_0$ is the conductivity, $\mathcal{A}$ is the cross-sectional area, and $s(\ )$ is another dimensionless function of the kind introduced in Eq. (10) [which is also computable using the expressions in Eqs. (6) and (7)]. Equation (12) defines $\delta V_\phi$ and $G_\phi$ (Refs. 4 and 9). When $\Delta E_F=0$, $|L_{tot}|=L_{in}$, and $G_\phi$ can be interpreted as the conductance of a piece of wire which is one phase-breaking length long.

The resulting statement of universality for the "naive" conductance $G\equiv I/V_{ij,kl}$ is

$$\delta G = \frac{e^2}{h}\frac{\langle G\rangle^2}{G_\phi^2}t(\ ) , \quad (13)$$

where again $t(\ )$ is the same kind of dimensionless function of order unity which has been introduced above in Eqs. (10) and (11). Equation (13) differs from the original statement of universal conductance fluctuations[1] in the presence of the additional factor $\langle G\rangle^2/G_\phi^2$, where $\langle G\rangle$ is the average conductance of the sample being measured. If this conductance is large compared with $G_\phi$, say because the voltage leads are closer together than $L_{in}$, then the fluctuations in the total conductance can become large. However, this is simply an artifact of using this "naive" conductance.

## III. CALCULATIONS AND RESULTS

We have computed various voltage fluctuations of interest in the two geometries shown in Fig. 1. We use a quasi-one-dimensional approximation, where we assume that wires are much longer than they are wide, and that all wires have the same cross-sectional area $\mathcal{A}$. This is in accord with most of the experiments.[4,16] This means that we only need to solve a one-dimensional version of Eq. (8):

$$[(-id/dz+eA)^2+L_{tot}^{-2}]\tilde{d}(z,z',\Delta E,A)=\delta(z-z') . \quad (14)$$

In Eq. (14) we use a vector potential $A$ which is appropriate for an AB magnetic flux passing through the center of the ring. We assume that no flux penetrates the wire itself. (This means that in all of our calculations, the "correlation field"[1] $H_c$ is infinity.) The solution for the propagator $\tilde{d}$ is a linear combination of the form $X\exp(z/L_{tot})+Y\exp(-z/L_{tot})$. The constants $X$ and $Y$ in each segment of the ring or wire are determined by matching conditions: continuity of the propagator, current conservation, $\tilde{d}=0$ at the perfect leads, and the appropriate slope discontinuity when $z=z'$.

We work in a gauge where $A$ is given by $A(\theta)=\phi/2[\delta(\theta)+\delta(\theta-\pi)]$, where $\phi$ is the flux enclosed by the ring. The vector potential in this gauge corresponds to adding extra phases to the matching conditions at the junctions between the ring and the outside leads labeled $L_p$ in Fig. 1. We attach the ports 1, 2, 3, and 4 of the ring [Fig. 1(a)] and the wire [Fig. 1(b)] to perfect leads a distance $L_{lead}$ from the junctions of ports 1 and 2 (or 3 and 4). We make $L_{lead}$ several times $L_{in}$; the resulting voltage fluctuation is the same as it would have been had we let $L_{lead}\to\infty$. This has been emphasized lately by Hershfield and Ambegaokar.[12] The propagator $d(z,z')$ which appears in the flow function $\phi(z,z')$ is calculated similarly, but the solution for the propagator in each segment is given by $A+Bz$, since $d(z,z')$ contains no cutoffs.

The first calculation which we report is the symmetric versus the antisymmetric voltage fluctuation $\delta V_{13,24}$ for the ring in Fig. 1(a). The results are plotted in Fig. 2. The main features of this data, which have been noted in previous publications,[8,12–15] are that the symmetric voltage fluctuation grows like $(C_{ring}/L_{in})^{1/2}$ for large $C_{ring}/L_{in}$, while the antisymmetric voltage fluctuations remain length independent. We find that when $C_{ring}/L_{in}$ is near zero, the symmetric and antisymmetric fluctuations become equal, in agreement with Hershfield and Ambegaokar,[12] but differing by a factor of $\sqrt{2}$ from Isawa *et al.*[13]

Here we wish to recapitulate the reason for this behavior. To do this, it is useful to introduce an equivalent way of computing the voltage fluctuation originally due to Maekawa *et al*:[11]

$$\langle \delta V_{ij,kl}^2\rangle = \frac{I^2}{\sigma_0^4}\int_I d\mathbf{r}_1\int_V d\mathbf{r}_2\int_I d\mathbf{r}_3\int_V d\mathbf{r}_4\Gamma^{ss,aa}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4) . \quad (15)$$

The $I$ and $V$ denote integrals along the current $(ij)$ and voltage $(kl)$ paths. We proved[8] that Eq. (15) is equivalent to Eq. (4) for the wire [Fig. 1(b)]. A similar expression has been derived by Isawa *et al.*[10] for the ring. Because of the functional form of the propagator $\tilde{d}$ above, the function $\Gamma(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)$ [Eqs. (6) and (7)] is small unless $\mathbf{r}_1$ and $\mathbf{r}_2$ are within an inelastic scattering length $L_{\text{in}}$; the same is true for $\mathbf{r}_3$ and $\mathbf{r}_4$. This says that the important regions of the sample in Fig. 1 for the integrals in Eq. (15) are those for which the current path and the voltage path are separated by less than $L_{\text{in}}$; this is true in the shaded region in Fig. 3(a).

For the antisymmetric response, however, there is another restriction.[13] Because of the fundamental Onsager symmetry[9] $\sigma(\mathbf{r}_1,\mathbf{r}_2,H)=\sigma(\mathbf{r}_2,\mathbf{r}_1-H)$, it is true that $\sigma_a(\mathbf{r}_1,\mathbf{r}_2)=-\sigma_a(\mathbf{r}_2,\mathbf{r}_1)$, and $\Gamma^{aa}(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3,\mathbf{r}_4)$ is antisymmetric with respect to interchanging $\mathbf{r}_1$ and $\mathbf{r}_2$, or $\mathbf{r}_3$ and $\mathbf{r}_4$, at least when $\Delta\mathbf{A}=0$ in Eq. (7). This means that the part of the integrals in Eq. (15) where $\mathbf{r}_1$ and $\mathbf{r}_2$ (and $\mathbf{r}_3$ and $\mathbf{r}_4$) are inside the region common to the current and voltage paths will cancel out because of the antisymmetry of the integrand. (This does not apply to $\Gamma^{ss}$, since the integrand in this case is symmetric.) Therefore, for $\Gamma^{aa}$ the relevant part of the sample does not include the region of the sample common to the current and voltage paths, only including the region of the sample within one $L_{\text{in}}$ of the junction between the current and voltage paths as shown in Fig. 3(b). Thus, the antisymmetric piece is independent of the length of the device common to the voltage and current paths, i.e., to the distance between voltage leads.

We have computed the amplitude of the AB oscillation measured in the four-lead voltage $V_{14,23}$. We do this by computing the correlation function $F(\Delta\mathbf{A})\equiv\langle\delta V_{14,23}(\mathbf{A})\delta V_{14,23}(\mathbf{A}+\Delta\mathbf{A})\rangle$, where the vector potential $\mathbf{A}$ corresponds to an Aharonov-Bohm flux $\phi$ passing through the ring. Then the Fourier transform

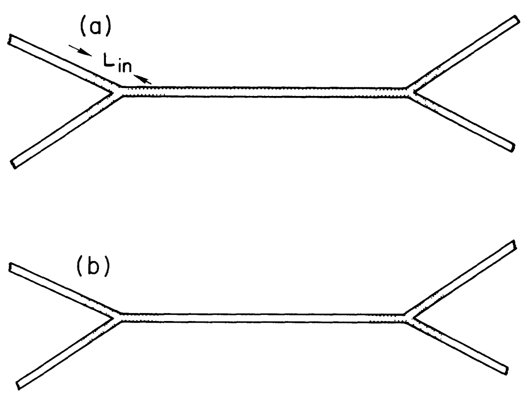



FIG. 3. (a) Shaded to show the portion of the wire device where the integration variables of Eq. (15) are relevant for the symmetric response. (b) The portion of the wire device where the integration variables of Eq. (15) are relevant for the antisymmetric response. Only the region within one $L_{\text{in}}$ of the junction of the voltage and current leads is relevant. The same is true of the ring.

$$\left[\frac{1}{2\pi}\int_0^{2\pi}F(\Delta\phi)e^{i\Delta\phi/\phi_0}d\Delta\phi\right]^{1/2}$$

gives an estimate of the amplitude of the AB oscillation in the four-point measurement. Here $\phi_0\equiv h/e$; it has been shown that $\phi_0$ (rather than $\phi_0/2$) should be the dominant period in these systems.[30–32]

Figure 4(a) shows the result of this calculation for both the symmetric and the antisymmetric voltage responses. Physical arguments suggest[27] that these fluctuations should decay exponentially with $C_{\text{ring}}/L_{\text{in}}$, since one might expect that only those carriers which retain phase coherence (i.e., do not scatter inelastically) while circling the ring will contribute to the AB oscillation. Indeed, asymptotically as $C_{\text{ring}}/L_{\text{in}}\to\infty$ we find that both the symmetric and antisymmetric oscillations asymptotically decay exponentially, with the antisymmetric decaying slightly faster than the symmetric. Büttiker[5] has obtained this result earlier using simple arguments. However, as the graph shows, one must go to rather large $C_{\text{ring}}/L_{\text{in}}$ before the curves approach this asymptotic result, so that when $C_{\text{ring}}\approx L_{\text{in}}$, a condition achieved by some experiments,[27] they are not accurately described by a single exponential.

In order to compare more closely with the experiment of Milliken *et al.*,[27] we have plotted in Fig. 4(b) the same AB oscillation as it would appear in the "naive conductance" which as used by the experimentalists is $G=I/V_{ij,kl}$. Again we see that the curves behave asymtotically like exponentials. The experiments[27] operate in the range $1.5\lesssim C_{\text{ring}}/L_{\text{in}}\lesssim 3$; we see that in Fig. 4(b), data over this narrow range could be interpreted in terms of a single exponential; but, the value of this exponent will not represent the true asymptotic behavior. The effective exponent is larger at small $C_{\text{ring}}/L_{\text{in}}$.

All of the above is in a regime where we assume that the magnetic field is much greater than zero, which allows us to ignore the particle-particle contribution to $\Gamma$ above. Now we would like to understand the behavior of the AB oscillations in the vicinity of zero field. In particular, experiments[17] were focused[5] on the phase shift of the AB oscillation with respect to the zero magnetic field axis. This phase shift was surprising because naively the voltage measurement was thought to be probing a diagonal Onsager coefficient, and the resulting Onsager symmetry would require that the AB oscillation should be symmetric around zero field and the phase of the oscillation should be 0 or $\pm\pi$. Of course, now we know that the four-terminal nature of the measurement means that the voltage fluctuation will have some antisymmetric part, so the phase need not be 0 or $\pm\pi$.

We extract the phase as follows: we expect the symmetric part of the voltage fluctuation to be of the form

$$\delta V^{s}_{13,24}(\phi)=B(H)+C(H)\cos(\phi/\phi_0)\ , \tag{16}$$

where $B$ and $C$ are slowly fluctuating quantities,[13] and $\phi_0$ is the fundamental AB flux quantum. We have ignored harmonics higher than the fundamental. Likewise, the antisymmetric voltage fluctuation will have the flux dependence[33]

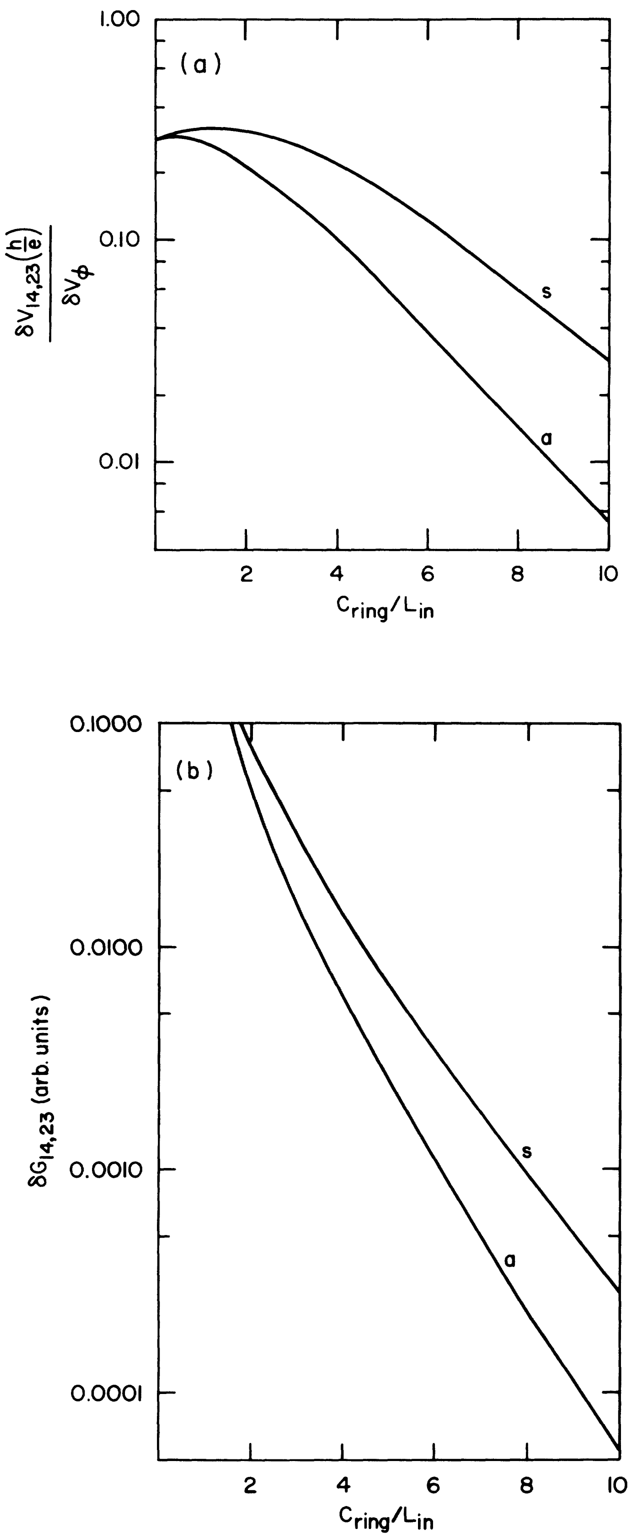



FIG. 4. (a) The symmetric ($s$) and antisymmetric ($a$) rms AB oscillations in $\delta V_{14,23}$ as described in the text. Both curves gradually roll over to an exponential decay, with the antisymmetric decaying a little faster than the symmetric. (b) Symmetric ($s$) and antisymmetric ($a$) rms AB oscillations in the naive conductance $G \equiv I/V_{14,23}$. We plot this because this is the quantity which is analyzed in Milliken *et al.* (Ref. 27). It too takes many $C_{ring}/L_{in}$ to reach an asymptotic exponential behavior.

$$\delta V^{a}_{ij,kl} = D(H)\sin(\phi/\phi_0) \ . \tag{17}$$

$D$ is also a fluctuating quantity. In order to obtain the phase, we need the relative sizes of $C$ and $D$. We can compute the root-mean-squared averages of $C$ and $D$ using the following correlation functions:

$$\begin{aligned}(\langle C^2\rangle)^{1/2} = \{ \tfrac{1}{2}[ \langle \delta V^{s}_{ij,kl}(\phi=0)\delta V^{s}_{ij,kl}(\phi=0)\rangle \\ - \langle \delta V^{s}_{ij,kl}(\phi=0)\delta V^{s}_{ij,kl}(\phi=\phi_0/2)\rangle ]\}^{1/2} \ ,\end{aligned} \tag{18}$$

$$(\langle D^2\rangle)^{1/2} = [\langle \delta V^{a}_{ij,kl}(\phi=\phi_0/4)\delta V^{a}_{ij,kl}(\phi=\phi_0/4)\rangle]^{1/2} \ .$$

Then assuming that the statistics of the probability distribution of $C$ and $D$ are Gaussian (suggested by certain recent numerical simulations),[34] we can obtain the probability distribution function of the phase $\eta$ of the AB oscillation $\delta V^{tot}_{ij,kl} = A\cos(\phi/\phi_0+\beta)$ through the relation $\beta = \tan^{-1}D/C$.

We show the results of the calculation of that probability distribution in Fig. 5. When $C_{ring}/L_{in}$ is very small, the symmetric and antisymmetric responses have equal amplitude (because, according to arguments given above, the region of the sample common to both the current and voltage paths is not significant in this case), so that the phase $\beta$ is equally likely to have any value between $-\pi$ and $\pi$, as shown in the figure. As $C_{ring}/L_{in}$ increases, there is a gradual preference for the phase $\beta$ to be near 0 or $\pm\pi$, the "classical" values. However, this preference develops rather slowly; as shown by the figure, even for $C_{ring}/L_{in} = 5$, there is still a rather large probability for $\beta$ being significantly away from 0 or $\pm\pi$. In the experiments,[17] in which $C_{ring}$ is comparable to $L_{in}$, values of $\beta$ spanning the whole range were found.

Figures 6(a) and 6(b) show the results for the nonlocal voltage fluctuations $\delta V_{12,34}$ in a wire and in a ring.

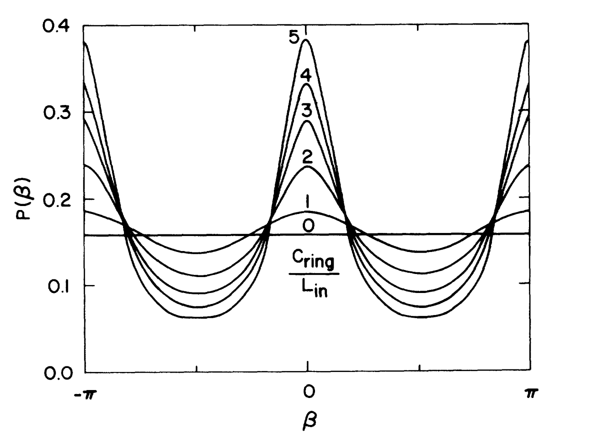


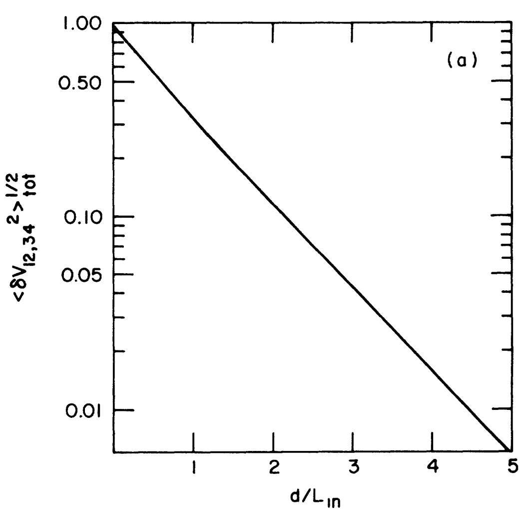

FIG. 5. The probability density function of the phase $\beta$ of the Aharonov-Bohm oscillation around zero magnetic field, as defined below Eq. (18), as a function of $C_{ring}/L_{in}$. For $L_{in}$ very large, $\beta$ is equally likely to be anything between $-\pi$ and $\pi$; when $L_{in}$ gets smaller the phase prefers the symmetrical values 0 and $\pi$.

"Nonlocal" means that the current and voltage paths do not intersect.[14,15] As mentioned in the Introduction, both of these fluctuations decay away exponentially; we find $\delta V^{\rm rms}_{12,34} \propto \exp(-\beta C_{\rm ring}/L_{\rm in})$, $\beta \approx 0.6$ for the ring, and $\delta V^{\rm rms}_{12,34} \propto \exp(-\alpha d/L_{\rm in})$, $\alpha \approx 1.1$ for the wire for the range of $d/L_{\rm in}$ studied. (One can show that when $d/L_{\rm in} \to \infty$, $\alpha = 1$.) The curves are well approximated by single exponentials even for very small $d/L_{\rm in}$; this does not contradict the slow power-law decay extracted from the simulations of Baranger *et al.*,[35] since their work contained no inelastic scattering, but did have perfect leads attached rather close to the sample.

Recent experiments[26] see this exponential decay in wires with an $\alpha$ fairly close to ours. We find that for a nonlocal measurement on either rings or wires, the antisymmetric fluctuations decay away with exactly the same exponent as the symmetric; this also seems to be consistent with experiment.[26] This equivalence of symmetric and antisymmetric responses means that the phase of the AB oscillations mentioned above can take any value between $-\pi$ and $\pi$ with equal probability in this nonlocal measurement. The AB amplitude decays away exponentially with the same decay constant as the rms fluctuations do.

In Fig. 7 we have cross correlated the fluctuations of two different voltage measurements, normalized to the total fluctuations of one of them. In these two different measurements, the role of one pair of voltage and current leads has been swapped. These measurements give identical average voltages, but the extent to which the fluctuations should be the same in these measurements is related to the extent to which the relevant parts of the sample (Fig. 3) for these two measurements are the same. For the symmetric voltage fluctuation, the two measurements have the sample segment between the two voltage leads in common [Fig. 3(a)]. When $L_{\rm in}$ is small, this common segment is most important, and the fluctuations are strongly correlated. When $L_{\rm in}$ becomes large, a larger part of the leads becomes relevant. The current and voltage paths in Eq. (15) become increasingly different, and the correlations become smaller. Benoit *et al.*[17] have seen a strong correlation between the symmetric voltage measurements done with different lead configurations, but they did not examine the $L_{\rm in}$ dependence of this correlation. On the other hand, for the antisymmetric fluctuations, the segment common to the voltage and current paths is not important, and the two voltage measurements are not correlated.

The Fermi energy correlation function of the voltage fluctuations is important because it determines an important component of the temperature dependence of the



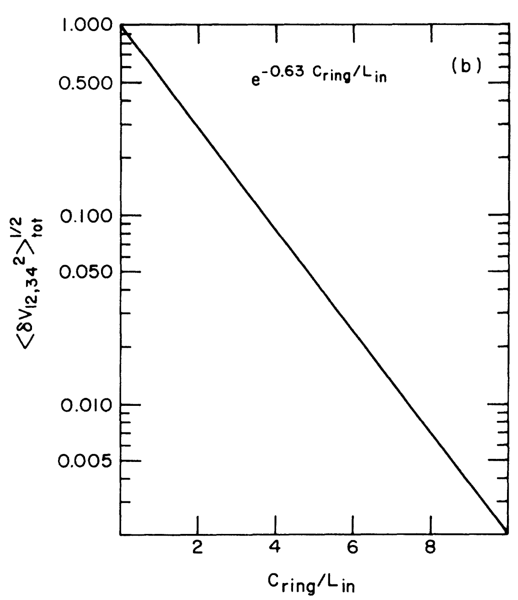




FIG. 6. (a) Nonlocal voltage fluctuation $\langle (\delta V^s_{12,34})^2 \rangle^{1/2}$, measured in units of $\delta V_\phi$, for the wire as a function of $d/L_{\rm in}$. [See Fig. 1(b).] The result is quite accurately fitted to an exponential as shown. (b) The same for the ring.

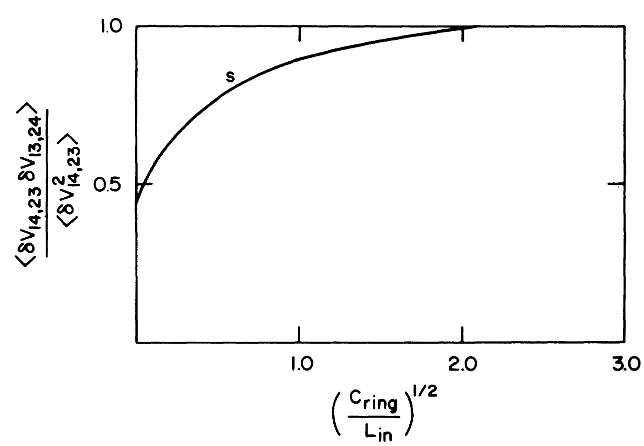



FIG. 7. The normalized cross correlation $\langle \delta V_{14,23} \delta V_{13,24} \rangle / \langle \delta V^2_{14,23} \rangle$. There is already a strong correlation for $L_{\rm in} \to \infty$, and the correlation becomes perfect as $L_{\rm in}$ gets smaller. This is true of the symmetric response; the antisymmetric responses is completely uncorrelated. This result is in agreement with a recent experiment (Ref. 17).

voltage measurement. It does this through "energy averaging,"[1] that is, the fact that at finite temperature not all the carriers are transported exactly at the Fermi energy, but rather within a window of width approximately $kT$ around it. Therefore, it is important to know how the voltage or resistance varies within an energy of $kT$; this resistance will be correlated within some energy $E_c$. The temperature dependence that this adds to the voltage fluctuations will be $(E_c/kT)^{1/2}$.[1]

We first compute the energy correlation function[1,15] Eq. (11) without any AB magnetic flux. Typical curves for these correlation functions are shown in Fig. 8. Several of the properties of these curves have been predicted previously by Lee *et al.*[1] First, it was predicted that these curves should fall off like $(C_{\rm ring}^2/\pi^2 L_E^2)^{-3/2}$, and we indeed find this to be the case. Second, it was predicted that $E_c$ should behave like $\hbar D/L_{\rm in}^2$, and we confirm this. We extract $E_c$ by taking the half-width at half maximum of the curves in Fig. 8; these are plotted in Fig. 9. The results are a straight line on this log-log plot, indicating a power-law relation $E_c C_{\rm ring}^2/\pi^2\hbar D \propto (C_{\rm ring}/L_{\rm in})^\alpha$. We obtain $\alpha \simeq 2.1$, which is consistent with the Lee *et al.*[1] result of $\alpha = 2$. We find that the same is true for both the symmetric and the antisymmetric voltage fluctuations.

We have also computed the energy correlation function in the presence of an AB flux. For each energy difference in the desired voltage fluctuation correlation function of Eq. (11), we Fourier transform in the magnetic flux difference:

$$f(\Delta E_F) = \frac{1}{2\pi}\int_0^{2\pi} e^{i\Delta\phi/\phi_0}\, d\Delta\phi \times \langle \delta V_{13,24}(L_{\rm in}, E_F, \phi) \times \delta V_{13,24}(L_{\rm in}, E_F + \Delta E_F, \phi + \Delta\phi)\rangle \,. \quad (19)$$

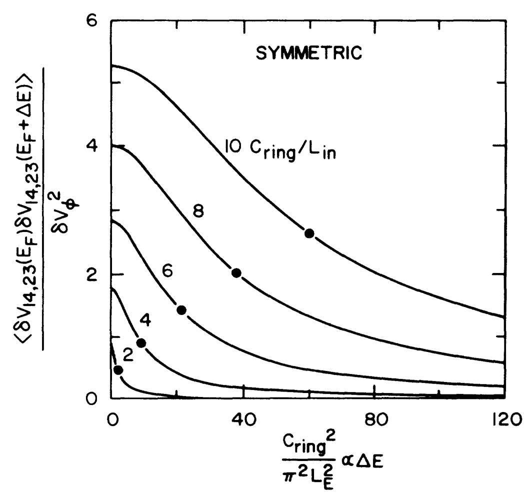



FIG. 8. The energy correlation function for the symmetric part of the voltage fluctuation $\delta V_{14,23}$ in the ring, for different values of $C_{\rm ring}/L_{\rm in}$ as indicated. These curves fall off asymptotically like $\Delta E^{-3/2}$, as previously predicted (Ref. 1). The heavy dots are the half-width at half maximum for these curves, which we take to be $E_c$, the energy correlation length.

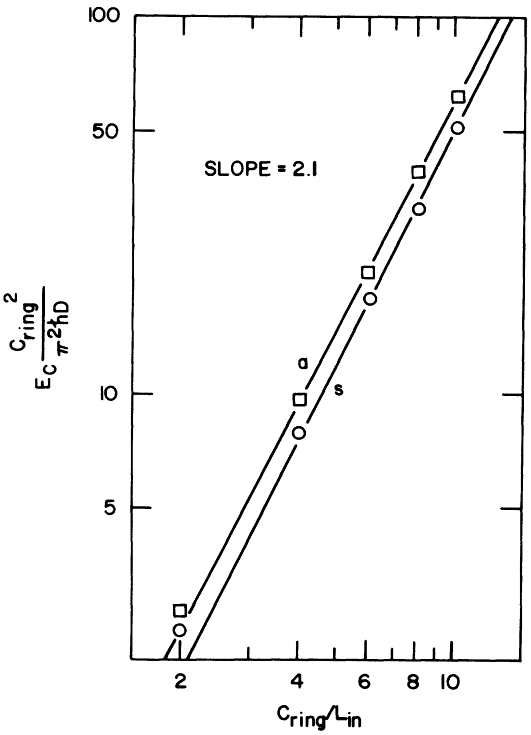



FIG. 9. The normalized energy correlation $E_c$ for the symmetric (circles) as well as the antisymmetric (squares) voltage fluctuation $\delta V_{14,23}$. The slope of about 2.1 is consistent with $E_c \propto L_{\rm in}^{-2}$, as predicted previously (Ref. 1).

We will ignore the particle-particle piece, so there is no dependence on **A** itself. In Fig. 10 we show the typical behavior for this $f(\Delta E)$ for different values of $C_{\rm ring}/L_{\rm in}$. These curves go slightly negative: the line shape is quite different from the energy correlation functions for the aperiodic conductance fluctuations in Fig. 8. Nevertheless, we can again use the half-width at half maximum of $f(\Delta E_F)$ to extract an $E_c$ which is appropriate for the temperature dependence of the AB oscillations. In Fig. 11 we present these in a log-log plot. Again the symmetric and antisymmetric contributions are about the

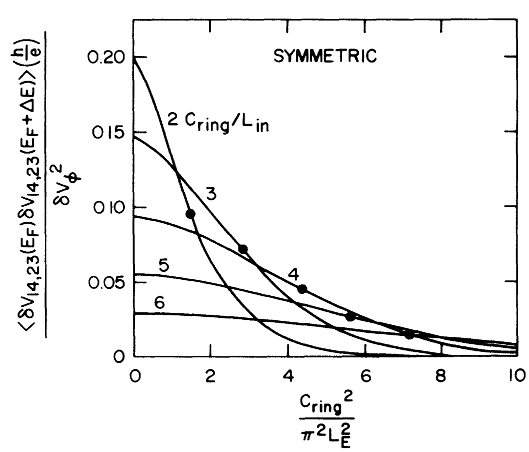



FIG. 10. The same as Fig. 8 for the AB oscillation in $\delta V_{14,23}$.

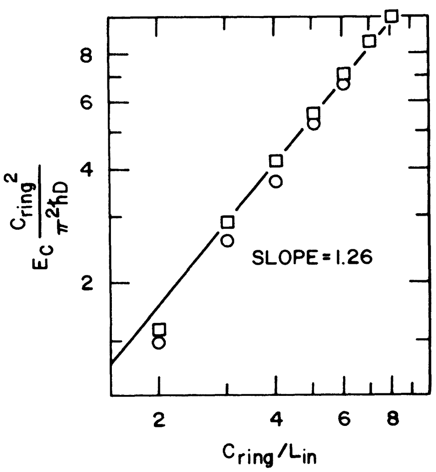



FIG. 11. The same as Fig. 9 for the AB oscillation in $\delta V_{14,23}$. The slope of 1.26 is consistent with $E_c \propto C_{ring}^{-0.7} L_{in}^{-1.3}$, we give arguments in the text that this should be $E_c \propto C_{ring}^{-1} L_{in}^{-1}$.

same; the power extracted in this case, however, is about 1.3. This means that $E_c$ has the dependence mentioned in the Introduction: $E_c \simeq \hbar D / L_{in}^{\alpha} C_{ring}^{2-\alpha}$, $\alpha \simeq 1.3$.

This result is different from that quoted in Isawa *et al.*;[10] their claim amounts to $\alpha = 2$ above. They give no argument for this answer. This result is also at variance with the argument given by Milliken *et al.*[27] They reasoned that the relevant time which determined $E_c$ was the typical time for a carrier to diffuse around the ring, given by $\tau^{-1} \simeq D / C_{ring}^2$, i.e., $\alpha = 0$.

Here we present a refinement of Milliken *et al.*'s argument. Most of the electrons diffusing around the ring are inelastically scattered before they can travel around; their survival probability is only $\exp(-C_{ring}^2 / L_{in}^2)$. On the other hand, there are atypical electrons which diffuse around the ring more quickly; there are fewer of them, but they have a much greater probability of making the circuit without being inelastically scattered. We can write the probability $p(t)$ that a carrier interferes with itself after one passage around the ring at time $t$ as the product of the probability that it has diffused all the way around, times the probability that it has not been inelastically scattered:

$$p(t) = e^{-C_{ring}^2/Dt} e^{-t/\tau_{in}} = e^{-(C_{ring}^2/Dt + Dt/L_{in})} . \tag{20}$$

Now, the typical time is approximately determined by the maximum of this $p(t)$, which is approximately where the term in the exponent is minimized. The result is

$$E_C \simeq \frac{\hbar D}{L_{in} C_{ring}} , \tag{21}$$

which corresponds to $\alpha = 1$ above. This is in rough agreement with the $\alpha = 1.3$ which we extracted above. The small discrepancy could arise from any number of sources: the fact that we are not strictly in the asymptotic regime of $C_{ring}/L_{in} \to \infty$, the crudeness of the above estimate of the typical time, and so on. Still, this result suggests that a new analysis of the experiments in Milliken *et al.*[27] would be in order.

## IV. CONCLUSIONS

These calculations have yielded new information of qualitative value for our understanding of mesoscopic rings and wires. We have confirmed the idea that for the symmetric voltage fluctuations, the region of the sample between the voltage leads controls the transport, while for the antisymmetric response, only the regions around the junction of the voltage and current probes are important; these are confirmed in Fig. 2.

We have verified that the AB oscillations are not symmetric around zero magnetic field, i.e., that classical Onsager arguments do not apply to the voltage measurement in the quantum regime. As $L_{in}$ gets smaller, the classical symmetry begins to apply.

We have verified the existence of nonlocal voltages, that is, voltages measured across a path which classically carries no current. As expected, these voltages are exponentially small if the nonlocal distance is greater than $L_{in}$.

Also as expected, the Aharonov-Bohm oscillations are exponentially damped by inelastic scattering (Fig. 4). Contrary to previous expectations, however, this result is asymptotically correct only for $C_{ring}/L_{in} \gtrsim 3$.

We have confirmed the correspondence between the energy correlation $E_c$ and the phase coherence time for electrons. However, it was not previously appreciated that for the AB effect, the relevant electrons are the atypical ones which diffuse very fast around the ring and do not lose phase memory.

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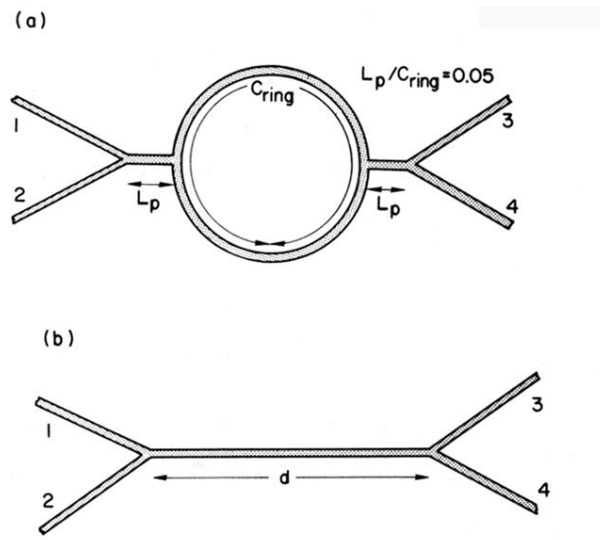


FIG. 1. (a) The geometry of the ring which we study in our calculations. $L_p$ is the distance from the ring to the voltage probes, and we take it to be about 0.05 of the circumference of the ring, $C_{\text{ring}}$. This is the experimental geometry used by Benoit *et al.* (Ref. 17). With our approximations, all the wires are quasi-one-dimensional, and the angle between the voltage and current wires is irrelevant. (b) The geometry of the wire studied in our calculation.

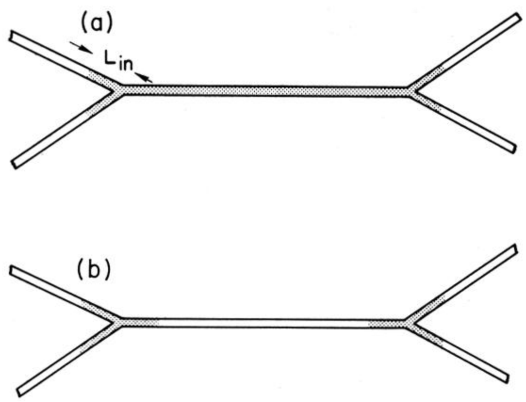


FIG. 3. (a) Shaded to show the portion of the wire device where the integration variables of Eq. (15) are relevant for the symmetric response. (b) The portion of the wire device where the integration variables of Eq. (15) are relevant for the antisymmetric response. Only the region within one $L_{in}$ of the junction of the voltage and current leads is relevant. The same is true of the ring.