Key Ideas about Quanta

The time has come to fold together all the ingredients in our physics soufflé, including quantization, diffraction, interference, the uncertainty relation, the action principle, probability, wave equations, diffusion equations, and the basic facts about the structure of atoms as known by the 1920’s. We have had a lot of foreshadowing of a need to revise classical physics, including:

(i) The need to fix the size of atoms.
(ii) The need to have stable atoms despite the mobile charges known to be inside them.
(iii) The need to understand the sharp spectral lines of atomic emission...
(iv) ... and the specific heats of gases, of radiation in a hot cavity, and of solids at very low temperatures.
(v) [There were also some paradoxes with the conduction of electricity through metals. For instance, why do some solids conduct, whereas others do not? One can even convert a solid from insulator to conductor by applying pressure!]

These were all historical problems early in the century. In your text you’ll read about the photoelectric effect and electron diffraction, two more key historical problems which got people thinking in the right direction. But first I’d like to present you with a more modern puzzle which puts the basic ideas of quantum theory in the sharpest possible relief. The experiments I describe are idealized, but essentially equivalent ones can be and have been performed, at least in part. Historically they weren’t the first experiments, but the historical approach isn’t always the best way to learn.

Looming behind the above list is one more question:

• How do we fix up these failures of classical physics without trashing its incredible successes? We need revolution, but a conservative revolution.

1. The electron experiment

The following diagram symbolizes the setup we want to study:
A hot filament at negative potential $V_0$ gives off electrons which get accelerated toward a grounded electrode, much as in Crookes’ cathode-ray tube. Some pass through a hole in the electrode and proceed through space to a screen with two slits. Then they land on an array of detectors. We can imagine the detectors as a lot of little separate plates, each of which can catch single electrons and amplify their charge to a detectable current. Then we build up an image from these pixels.

We notice that

1. The detectors register individual events if the current from the filament is low enough.
   
   We can crank down the current so low (or make the anode hole small enough) that we’re sure that at any given moment there’s at most one free electron in the apparatus. So all electrons passing through can be regarded as totally independent of the others.

2. The events are sharp in space and time. Each corresponds to depositing the fundamental charge $e$, and the kinetic energy $eV_0$, in just one detector. There are no fractional events, and only one pixel fires at at time.

So “obviously” we conclude that what comes off the wire are little lumps of charge localized in space and time which can be acted on by the accelerating potential $V_0$ just like macroscopic objects. Or in a word: bullets. We can also measure $e/m$ of these bullets, e.g. by comparing the transit time to the applied $V_0$; it agrees with the value estimated by Zeeman for the entities inside atoms that radiate light.

Unfortunately this tidy state of affairs begins to unravel when we notice the picture we get.

3. The pixels fire at random. There is a statistical pattern, but nobody has ever found a way to predict in advance which will fire next. The pattern slowly builds up as more and more electrons arrive at the detector. Here are some real data:

\[
\begin{align*}
\text{pin hole} & \quad \text{screen} & \quad \text{detector array} \\
\text{individual pixel}
\end{align*}
\]
The figure shows how photon arrivals build up a diffraction pattern at increasing exposures.

4. The pattern depends on the experimental geometry. If we close one slit we find diffraction. Unlike bullets, the pixels can fire off the line of sight; electrons can go around corners, like light or water waves.

5. If we open both slits we find an interference pattern, exactly like Young’s experiment with light!! In particular, opening the second slit can decrease the number of times a particula pixel fires, even to zero! There is no way this could happen with classical bullets. The pattern is the same as the one for waves of wavenumber $k$; where$^1$

$$k = \frac{p}{\hbar}, \text{ de Broglie's relation}$$  

$p = \sqrt{2m_e eV_0}$ is the electron’s momentum, and $\hbar$ is a new constant of Nature (the one discussed in “Units”).

In short

- Electrons are lumps. They do thin out after traveling long distances (i.e., fewer lumps per second land on a given area), but they retain their integrity; each lump has the same energy and charge no matter how far it traveled.
- Where any given electrons lands is unpredictable, but the pattern of where electrons land is statistically predictable in terms of a probability distribution $P(x)$.
- The probability $P_{ab}(x)$ of landing at $x$ with both slits $a$, $b$ open is not the sum $P_a(x) + P_b(x)$, but rather displays interference. More precisely, it behaves like the intensity $|\vec{E}|^2$ of light, not like the electric field $\vec{E}$. The latter keeps reversing its sign,

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$^1$ Some books phrases everything in terms of a different constant $\hbar$, defined as $\hbar = 2\pi\hbar$. Thus the wavelength is $\lambda = \hbar/p$. 
whereas the intensity, like our probability $P$, is by definition always positive. That is, the probability of arrival is a real positive quantity which acts like the square of some interfering wave amplitude.

2. Other experiments

In fact one could do the experiment all over again with light. Instead of the hot filament and accelerating potential, shine light; instead of the charge collectors put a very sensitive TV camera. Choosing single-color light (e.g. from a monochromater) and making the pinhole extremely small (to cut down the intensity) we find a few differences of detail. Mainly, the lumps now have no electric charge, and their energy instead of being $eV_0$ now follows the photoelectric rule:

$$E = \hbar \omega,$$

Einstein’s relation  \hspace{1cm} (2.1)

where $\omega = 2\pi/T$ is the angular frequency. Mostly, however, we see exactly the same thing as before:

1’, 2’: Individual events, localized in space and time.
3’: Pixels fire at random.

4’, 5’: But statistically they build up the familiar 1-slit diffraction and 2-slit interference patterns. That is, the probability $P(x)$ of arrival at $x$ gives the intensity of observed light.

Now of course 4’, 5’ are the expected features, whereas 1’, 2’ boggle the mind and contradict classical physics. Classical physics gives a picture of energy uniformly spreading out and diluting over larger regions as the wave leaves the slits. Instead, the experiment says that, like electrons, light too appears to be made of lumps. Let us call these lumps photons.$^2$

Note that combining the electromagnetic wave formulas $\omega = ck$ and $E = pc$ with (2.1) yields the de Broglie relation (1.1): Photons obey the same rule as electrons. Note too that a particle with $E = pc$ must be massless. We saw earlier that a massless particle can and must move at speed $c$. That’s what light does.

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$^2$ This gets back to the question of why we need sunscreen: It’s because the photons don’t smear out in space; although the probability to get a photon goes down like $1/r^2$, still each photon event involves the full energy $\hbar \omega$ getting released in a small region. For ultraviolet light (larger $\omega$ than visible light), this lump of energy is enough to break chemical bonds, and that’s why we need sunscreen.
Does this mean Maxwell’s celebrated equations are wrong? Well, at the very least they can’t have the same interpretation you learned in first-year physics. What about all that “wavelike” behavior you learned about? Feynman’s book QED shows how all the usual “wavelike” results of optics really do emerge from the photon viewpoint.³

Actually there is some conceptual tricky business with light (having to do with relativity), so nearly always in this course I’ll deal with electrons instead, and indeed electrons moving much slower than light. You showed in the homework that this is the situation inside atoms.⁴ One can go still further and do the experiment all over with other kinds of rays, for instance beams of neutrons selected to have all the same kinetic energy. Always we find the same rules! It’s crazy, but at least everything obeys the same rule.

Light isn’t a classical wave, and it’s not a stream of classical bullets. Whatever light is, it’s the same thing as electrons, neutrons, and indeed every fundamental particle ever discovered. In each case the particle character is related to the wave character by a formula involving the universal constant of Nature ħ (see the two boxed equations).

3. Four Proposed Laws

What physical law could we guess that correctly predicts the probability of arrival \( P(x) \) in all these similar cases? It looks bad. When two mutually exclusive routes are allowed to get a result \( x \) (e.g. electron seen at \( x \)), probability theory says that \( P(x) = P_1(x) + P_2(x) \). Definitely that fails here, just on experimental grounds. So we must conclude that passage via slit \( a \) or \( b \) are not exclusive possibilities. That’s not impossible; in water waves the fronts emerging from the two slits are correlated; they have the same phase, and so can interfere. But it’s bizarre: isn’t an electron much smaller than the slit separation?

In any case something is behaving additively. Our experience with classical waves, and the identification of intensity with probability, then suggest some provisional hypotheses:

1. The probability distribution of arrival at \( x, \) at time \( t, \) is the magnitude squared of some complex function \( \psi(x, t) \). That is, \( P(x, t)dx = |\psi(x, t)|^2dx \).

³ As so often happens in conservative revolutions, the key players reemerge later in new clothes. Though you don’t see them in Feynman’s book QED, Maxwell’s equations still play a key role in the quantum theory of light. (They determine the correct form of the action \( S \) to use.)

⁴ Remarkably, it’s also a reasonably good approximation for quarks in the proton. But there are other situations where things are moving fast, and for these we need the full machinery of quantum electrodynamics.
Remarks: Unlike the electric field, $\psi$ is a scalar quantity, not a vector (it doesn’t point in any spatial direction). But unlike $\vec{E}$, $\psi$ is complex, so it can still display interference.

II. $\psi(x,t)$ behaves deterministically: we can predict its future from its past, just as the diffusion equation lets us make precise statistical predictions about where particles are likely to be on average in the future. We call $\psi$ the wavefunction of the electron, photon, etc. In fact we’ll see eventually that $\psi$ is a solution to a wave equation, but not the same wave equation as the one describing light.

*The wavefunction is our new quantum concept of “state.”* It replaces the old classical definition of “state” as the values of position and velocity. We’ll have to work to understand this.

III. When one particle (electron, photon . . .) can create situation $x$ (some specified detector located at $x$ fires) in several ways, the wavefunctions, not the probabilities, add: $\psi_{ab}(x) = \psi_a(x) + \psi_b(x)$. (This is analogous to the fact that when a light ray can get to $x$ in several different ways, the amplitudes $\vec{E}$, not the intensities $\vec{E}^2$, add.)

Of course, I cannot overturn the laws of probability; when several different situations are possible and each excludes the other (e.g. the probability that either detector $x$ or detector $y$ fires, $x \neq y$), then the probability had better be $|\psi(x)|^2 + |\psi(y)|^2$.

I’m just saying that “passes via slit a” doesn’t necessarily exclude “passes via slit b.” For water waves this is obviously true; for Newtonian bullets it’s obviously not. For electrons and photons, apparently it’s true, since otherwise interference could not occur.\footnote{In fact if we do manage to modify our apparatus so that we’re sure which slit the electron went through, *then the interference goes away!* Nature just barely avoids a logical contradiction here.}

IV. Something must happen to every electron. So the distribution has to be normalized:

$$\sum_{\text{pixels}} |\psi(x)|^2 = 1, \quad \text{Normalization.} \quad (3.1)$$

Hypotheses (I–IV) are very radical, and we can’t accept them without a whole lot of experimental evidence. Most of the rest of this course is consecrated to this task. But they have stood the test of time. They are the basic quantum laws. You may not be so shocked by them because you’ve probably heard about them on the street, but take a minute to put yourself in the shoes of a 1920’s scientist — take a minute to be shocked. You can’t “understand” them; nobody ever has. “How does any given electron know that both slits
are open?" — the question has no answer. Our job is to frame guesses like (I–IV) and test them to see if they’re true and useful.

Historically, things weren’t helped by the fact that the probabilistic interpretation (the most radical part) wasn’t obviously necessary to explain the earliest experiments. Max Born, Pascual Jordan, and others slowly grasped that it was however logically necessary, the only way to reconcile the observed particle and wave character of light without logical contradictions. They also realized that many precise predictions (like energy levels) could nevertheless be made; we can live with this strangeness and still do physics.

What’s the origin of the indeterminacy in the quantum laws? Nobody knows. The deep physical laws are irreducible; they cannot be explained by appeal to some other, more intuitive ideas. Similarly Maxwell’s equations, including the mysterious displacement current term, are “just so.”

Anyway you can see that (II) is going to help us implement Bohr’s suggestion that the quantization of atomic energies, and in particular the stability of the lowest level, is like the quantization we found already in circular organ pipes.

But we have a long way to go still. For one thing, (II) is much too vague. We need a more specific law telling us how to compute $\psi$. And not just in free space! Electrons bound to nuclei are of course our greatest concern in atomic physics.

The incredible surprise is that laws (I–IV) turn out to be so simple and universal, even when we find the detailed form of (II)! They may be tricky to deal with in practice, but I’m going to tell you the answer right here. As more and more fundamental particles have been found, every one has turned out to obey these laws without modification.

As I keep emphasizing, this is the central miracle of physics — that laws of this simplicity and generality exist at all. That’s why physicists like physics.

Finally, for completeness I mention one more law with the same overarching generality as the others:

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6 For instance, the paradoxes (i)–(iv) above do not obviously point to any inability to predict the future in detail from the past, the experiment in §1 was technically too hard to do for many years to follow, etc. People were scandalized by the idea that they would have to give up strict, detailed, determinism, and confusion reigned for a long time.

7 Maxwell tried hard to explain the equations in terms of little gears and pulleys filling empty space, but it never came to anything. Again, our job is simply to make very precise guesses, and then test them. The good news is that, although the fundamental laws can’t be “explained,” at least there are very few of them, and they cover an immense spectrum of diverse phenomena.
\textbf{V.} Every electron is \textit{indistinguishable} from every other, even in principle. Every photon is indistinguishable from every other photon, every neutron, etc. There’s no way to “tag” one electron and keep track of it individually.

We’ll defer discussing \textbf{V} for a while. It won’t matter in hydrogen, where there’s just one electron, but already in helium it will have a profound effect. If \(\psi(x_1, x_2)\) is the wavefunction for one electron to be observed at \(x_1\) and the other to be observed at \(x_2\), it means that we must have

\[
|\psi(x_1, x_2)|^2 = |\psi(x_2, x_1)|^2, \text{ Exchange Symmetry. (3.2)}
\]

Wavefunctions not obeying (3.2) are just not allowed states of a 2-electron system. We’ll make this more concrete later on.

\section*{4. Three versions}

There are three equivalent ways to state the precise form of (II). We’ll be studying two of them in this course.

\textbf{IIa. Feynman’s law:} To compute \(\psi(x, t)\) pretend for a moment that electrons really are bulletlike objects. Consider every path an electron could take from 0 at \(t = 0\) to \(x\) at \(t = T\), passing through allowed regions (\textit{i.e.}, consider only paths passing via the slits). Consider even paths not obeying Newton’s laws. For each path compute the action \(S[\text{path}]\), and then the complex phase \(e^{iS[\text{path}]/\hbar}\). Add up all these numbers and multiply by some overall constant \(C\). This gives the wavefunction \(\psi(x, T)\). Choose the constant \(C\) in order to satisfy the normalization condition, (3.1). Thus

\[
\psi(x) = C \cdot \sum_{\text{all paths from 0 to } x} e^{iS[\text{path}]/\hbar}, \text{ Feynman’s law. (4.1)}
\]

That’s it.\footnote{This may look strangely familiar. Didn’t Boltzmann say that the probability of a state in classical statistical physics was \textit{also} given by an exponential weight function? This formal resemblance creates a deep link between statistical physics and quantum theory. On the other hand there are some big differences. Feynman’s weights \(e^{iS}\) are all \textit{phase factors} due to the factor of \(i\); in this sense every possible path gets the \textit{same} weight. In contrast, Boltzmann’s weights are all positive real numbers, not phases, and they’re \textit{not} all the same: configurations with lots of energy are very improbable.}
Notice how Feynman’s law is compatible with our law III: if there are two slits and every path must pass through one or the other, the answer can be written as the sum over all those passing through the first, plus all those passing through the second.

Why does the action enter here? We’ll see soon. What’s \( \hbar \)? Feynman says, it’s some new constant of Nature. It’s universal in the same way that Boltzmann’s constant \( k_B \) is: All kinds of objects share the same value of \( \hbar \). Clearly it must have the dimensions of action, \( [\hbar] = ML^2T^{-1} \). But these are also the dimensions of angular momentum, so the constant in Feynman’s law matches our police sketch of \( \hbar \) (see “Units” handout). Feynman says one new constant covers all situations, all particles, period.

Actually, a very different-looking form of III came first historically:

**IIb. Schrödinger’s law:** Get \( \psi(x, t) \) by solving a certain differential equation. This equation tells us \( \psi(x, t) \) in the future if we know \( \psi(x, 0) \) now. In this sense \( \psi(x) \) describes the new notion of “state.”

In a later lecture, I’ll show how to obtain Schrödinger’s equation from Feynman’s law. (Alternatively, your text motivates Schrödinger’s equation directly.) But we can see some features that such an equation must possess. For example, in order for IIb to be compatible with the addition rule III, the differential equation must be linear in \( \psi \). Unlike the diffusion equation, the equation we seek must also have traveling-wave solutions (after all, electrons can travel millions of miles through space!). The electromagnetic wave equation is linear and has traveling-wave solutions, but it’s not what we want: unlike light, electrons can move with many different velocities, not just one. Thus the desired equation must have dispersion, unlike the electromagnetic wave equation.

In fact, the Schrödinger equation will turn out to look a bit like the diffusion equation. But whereas the latter controlled the evolution of a positive real function (the concentration, or probability density of finding a particle somewhere), Schrödinger’s equation controls the evolution of a complex function, \( \psi \).

For completeness, I must mention that there is yet another (the third) equivalent approach to computing amplitudes, invented by Heisenberg. You’ll learn about it in your later QM courses.

5. First remarks

1. Whichever formulation you use, notice what’s going on: quantum theory, guided by the experiments discussed in §1–2, tells us we are never going to be able to say in detail which alternative possibility will happen in a given trial. That looks like a loss of predictive
power compared to classical physics, which says that knowing the initial conditions well enough lets us predict the future. Nevertheless, the probability distribution of events is well defined and precisely calculable.

Many people still think that the probabilistic character of the quantum laws is the key difference between quantum and classical physics. Actually, we now know that even classical systems with just a few degrees of freedom can give random time series; this phenomenon is called chaos. For instance think of balls dropping through a maze of pins. They end up at random positions, though with a definite probability distribution. The key difference between quantum and classical physics is the fact that quantum probabilities can display interference. This doesn’t happen in chaos. It’s what forced law (1) upon us.

2. The quantum laws also look superficially like the situation with diffusion, where the path of each molecule was unpredictable and yet the distribution obeyed a deterministic equation (which even looked a little like Schrödinger’s equation). The big difference is that then we were talking about a system composed of zillions of particles, whereas now we’re talking about just one electron. In the quantum world physics is only statistically predictive, in an even more fundamental way than in kinetic theory.

3. But we will be compensated for our loss of predictive power. Although certain things you may have thought you had a right to predict precisely will have to go, nevertheless other things you didn’t have a right to predict in classical mechanics (atomic energy levels, sizes) will come out of QM! So it’s not right to say “everything’s uncertain” as if this were a form of nihilism. It’s just that the definite predictions aren’t the things you thought they would be.

4. To get a feeling for what’s happening, suppose a particle moves freely (no forces) through space with momentum p (in our electron example, $p = \sqrt{2meV_0}$). The classical action is then $S = \frac{1}{2}mv^2T$ for an electron moving with velocity $v$ over a trajectory lasting time $T$. Rewriting in terms of the distance $x$ covered, $\frac{i}{\hbar} S[\text{path}] = \frac{im}{2} \frac{x^2}{T\hbar}$. Looks like at a nearby point $x + dx$, $\psi$ will differ by a factor of $e^{im(x/T)(dx/\hbar)}$. (I

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9 You may ask about the screen with the slit, the filament, etc. In practice they are complicated systems made of lots of atoms. In fact, these complications are not the root of the statistical character of quantum mechanics. Even if you have a single neutron just sitting in free space, there is absolutely no way to measure its detailed state and predict exactly when it will disintegrate (it splits into proton+electron+antineutrino). And yet there is a definite value for how long on average a neutron lives.
dropped the tiny \((dx)^2\) term.) So \(\psi\) oscillates as desired (see (ii)). The wavelength is
\[
\lambda = \text{distance to go through } 2\pi = \frac{2\pi\hbar}{p} \quad \text{or} \quad k = \frac{p}{\hbar} .
\] (5.1)

So we reproduce de Broglie’s relation, experimentally observed in the Davisson–Germer experiment.\(^{10}\)

\(\lambda\) works out to be mighty small for kilogram-type objects at ordinary speeds, due to the smallness of \(\hbar\). Even for a bacterium we can neglect the wave character. Only for extremely light objects, like individual electrons, will we notice \(\lambda\). Even then, the extreme smallness of \(\lambda\) is what enables electron microscopes to resolve tiny submicron objects (like viruses). Only for \textit{atomic dimensions} will the wave character of electrons matter, but then it matters a lot.

A very important point about (5.1) is that according to our laws it’s universal. It contains no constant like the charge or mass that depends on what kind of particle we have, just \(\hbar\), which we claim is universal. This is easy to check by comparing the diffraction patterns of electrons to those of neutrons, say, and it works: \textit{the wavenumber divided by the momentum is a universal constant}.

5. Suppose instead we sit at one place \(x\) but let \textit{time} go by. That is, compare the transmission to \(x\) in time \(T + dT\) to transmission in time \(T\). This time we get a change by a factor of \(e^{i\hbar x^2 / (2\pi)} dt = e^{-iEt/\hbar}\). So \(\psi\) oscillates in time too, like a good wave should, with a frequency given by Einstein’s formula
\[
\omega = E/\hbar .
\] (5.2)

Again we note that this formula is universal: \textit{the frequency divided by the energy is again a universal constant}.

6. What if the electron \textit{isn’t} free, but feels forces? We’ll get back to it, but if it moves in a \textit{constant} potential \(V_1\) then the action just changes by \(-V_1 T\) and we again get (5.2), this time with the total energy \(E\) and not just the kinetic energy.

6. \textbf{Vistas}

We can begin to see what’s going to happen to our initial puzzles (i–iv). Starting from (iii), now that we know that the color of light is related to the \textit{energy} of its photons,

\(^{10}\) You’ll see in a homework problem that this conclusion isn’t changed when we sum over all the paths, instead of just taking the classical path. Also see Feynman & Hibbs §3-1.
sharp spectral lines consist of photons of definite energy. They must get released when atoms go from an excited state of high, but definite, energy, to a less excited state of lower, but definite energy. So the challenge becomes, how to explain the fact that atoms have states of definite energy? We’ll answer this soon, but you can imagine already that it’s got to do with quantization in organ pipes. Once we let a wave property into the game we have to expect quantization whenever that wave gets confined to a region comparable to its wavelength. Then the frequency, like the frequency of a violin string, will have to take on one of a set of discrete values, and hence the energy will, too.

Feynman’s law is still not so easy to use. Nor is it at first clear how it reproduces all the results of classical physics when applied to big things, like the Moon. After all, it says to give equal weight to every possible path, since $|e^{iS}| = 1$ always! Why then does the Moon choose the path obeying Newton’s law?

We get a hint from the diffraction of light. Consider a single, fat slit. Every path through that slit gives a contribution to the total, and all the contributions have the same magnitude. But the phases conspire to wipe out the signal except along a straight-line path, if the slit is fat (see the “Diffraction” handout). Mathematically, what’s special about the line of sight is that this is the path minimizing the transit time of the light ray. Indeed, in the limit of macroscopic apparatus or very short wavelength, light obeys ray optics, where it travels on straight lines minimizing the transit time (see “Action” handout).

That’s what we’d like for other particles, too. The Moon has a very short wavelength, and so according to Feynman’s rule it is overwhelmingly likely to be found on the path minimizing the action. But the path minimizing the action is just the path obeying Newton’s law! Even for electrons something similar happens in a (meter-sized) laboratory, which is why Thomson’s electrons behaved like little charged bullets. Only on distances comparable to the electron wavelength will the wave character matter.

Thus ultimately the smallness of $\hbar$ is what makes classical physics work for everyday life.\footnote{By the way, this is also the deep reason why $\hbar$ has to be universal. Suppose neutrons and electrons somehow had different values of $\hbar$. Then the phase factor $S_{\text{neutron}}/\hbar_{\text{neutron}} + S_{\text{electron}}/\hbar_{\text{electron}}$ wouldn’t have its minimum at the same path as the classical $S_{\text{neutron}} + S_{\text{electron}}$ and we wouldn’t recover the correct classical limit.}

7. Summary

We’ve been led to a picture in which electrons are neither particles, nor waves! Instead
All fundamental entities (electrons, photons, neutrons...) interact locally, at one point. But we have no way to predict in advance exactly which place we’ll find an electron. All we get is probabilities, and these probabilities have a wave character that becomes evident whenever the distances in question are smaller than the de Broglie wavelength.

We’ve been led to propose that a state of a 1-electron system is not specified by a position and a velocity, but rather by a complex function $\psi(x)$. The wavefunction $\psi$ is quite different from the electric and magnetic fields of classical electromagnetism. For instance, in classical physics it was no problem to measure $\vec{E}$ directly: just put in a small test charge and measure the force on it. But I haven’t given you a method to measure $\psi(x)$ directly, and I never will. Nobody has ever found a way to do this. Instead $|\psi(x)|^2$ yields the probability to find an electron at $x$. We can measure this to any accuracy just by repeating a measurement many times with identical initial states. But we can’t measure $\psi$ itself directly.

Many people found this situation disgusting. How can we do serious science with quantities we can’t measure? Isn’t this edging towards astrology? But in the end this is just another case of how to discuss things you can’t see. We can do it if we find an interlocking web of many different experiments all of which are best explained in terms of this indirectly-observed construct. It’s risky, but we have to take the risk. Nothing else has worked. Moreover, the alternative, classical measurement theory, turned out to be not so well grounded after all, as we’ll see in a moment.

The interpretation of $\psi(x)$ rests on a subtle point. Certainly there is a state which, at time zero, is known to be located near $x = 0$ to great accuracy. This state is described by a wavefunction $\psi(x)$ which is zero nearly everywhere, with a narrow bump near $x = 0$. But what about momentum? When I send an electron through a velocity filter, I certainly know its momentum to very great accuracy, and yet I don’t see $p$ anywhere in this wavefunction that purportedly tells me all about the state. The answer is de Broglie’s relation: momentum is related to the wavelength of the wavefunction.

8. Uncertainty

You may be forming an objection. Think back to the “Wave Equation” notes. There we saw that a wave can be sharply localized in space, or it can be a pure tone of one wavelength, but not both at the same time.
• Indeed, a pure sine wave has a definite wavelength, and hence by de Broglie a definite
momentum, but it goes on forever — it’s not located anywhere in particular.
• Conversely, to synthesize a sharp bump we had to mix together a wide range of
wavelengths. The corresponding state \( \psi(x) \) must describe an electron with a well-
defined position but a poorly-defined momentum.

What do we mean by “poorly-defined momentum”? Simply that when this electron inter-
acts it will deliver a definite kick, but before it interacts it’s impossible in principle to say
just how big that kick is going to be: Again all we get is probability.

What we found was that the product of the spread \( \Delta x \) in position times the spread
\( \Delta k \) in wavenumber was always a bit bigger than one (remember, \( \Delta x \Delta k \) is dimension-
less). This was a Merely Mathematical fact about waves. But de Broglie’s interpretation
of wavenumber as momentum now gives this a physical meaning: I can prepare a state
localized in space (confine the electron to a tiny box), or with definite momentum (put it
through a velocity filter), but the product of the uncertainties always exceeds \( \hbar \):

\[
\Delta x \Delta p \geq \hbar , \text{ First Uncertainty Relation} \tag{8.1}
\]

We can use our Merely Mathematical fact in the time domain as well. By exactly the
same logic, a wave can either have a well defined frequency, but last a long time, or be
very short but contain many pitches. In the former case we can’t say exactly “when” the
sound happened: its uncertainty \( \Delta t \) is large. Thus \( \Delta t \Delta \omega \geq 1 \). To give this mathematical
fact about waves a physical meaning, we just need to remember Einstein’s interpretation
of the frequency! Thus

\[
\Delta t \Delta E \geq \hbar , \text{ Second Uncertainty Relation} \tag{8.2}
\]

Heisenberg came upon both of these uncertainty relations in a very different way; it
took some time to realize that they were just consequences of the wavefunction approach
and the properties of waves. Notice that these limitations on what we can measure are
totally invisible in everyday life. Normally we measure things to the nearest millimeter,
and their momentum to similarly large error. So normally we never come anywhere close
to the precision needed to feel these limitations.

Try this: Think of a bacterium as a water-filled sphere about 1 \( \mu \)m in radius. Suppose
we try to measure its speed to an accuracy of 1 \( \mu \)m/s, and its position to an accuracy of
10% of its radius. Do we run into any problem with the uncertainty principle?
But does this really answer the objection? Should we really be content with a situation where we can’t know the position and momentum, even in principle? Heisenberg thought about this for a long time. The answer turned out to hinge on a close analysis of what experiments could really be done. Heisenberg asked, if classical mechanics wants to say that a state is specified by position and momentum, can we really measure these together to perfect accuracy? He imagined a very powerful microscope which could pin down the location of an electron to very high precision. Well, that requires a microscope using light of very short wavelength. But such light carries a tremendous kick. When it bounces off an electron we indeed find the latter’s position, but its momentum gets all messed up. And moreover, it gets messed up by an amount inversely proportional to the photon wavelength, exactly as predicted by (8.1)! In the previous section I mentioned that some found it distasteful to base a theory on a notion of “state” which contained a wavefunction $\psi(x)$ you couldn’t measure directly. But now we see that classical physics is founded on an equally unmeasurable notion of “state”! There’s no purely aesthetic way to choose one over the other.

After many thought experiments, Heisenberg and others finally concluded that

Classical physics is resting on the hidden assumption that perfectly accurate measurements of $x$ and $p$ can be made. Since this just isn’t true, we are free to abandon this assumption, free to entertain a theory where $x$ and $p$ cannot be measured to perfect accuracy. Quantum mechanics is such a theory.

Of course we’re not required by logic to abandon classical physics. It’s experiment which is forcing us to do that (remember our growing list of experiments not explainable by classical mechanics). Heisenberg just pointed out that the apparently radical ideas of quantum mechanics were not obviously inconsistent.

9. Payoff

We just saw in two different ways that uncertainty doesn’t keep us from applying classical physics to macroscopic objects: in §6 I said it in terms of the short-wavelength limit of optics, and in §8 I said it in terms of the uncertainty relation. At the other extreme, for an electron in an atom we can expect classical physics to be a hopelessly bad approximation. That’s good! We don’t want atoms to radiate and collapse, as classical physics says they must! Let’s just get a back-of-the-envelope estimate of how big atoms should be using our new ideas.
First of all, if the wavefunction $\psi(x)$ is very sharply peaked near one point, then it’s overwhelmingly likely that in this state a measurement will find the particle there. But the uncertainty relation then implies that $\psi$ is a combination of many different waves with a big spread in wavenumber $k$. And then de Broglie’s relation means that the momentum of the electron is poorly defined. That is, if we try to measure the momentum in this state we’ll get a lot of different answers with a big variance $\Delta p$.

If an atom is too small then the electron is confined to a tiny size $a$ in space. But by the uncertainty relation (true for any kind of wave), this means its wavefunction is a combination of many sine waves, with a spread in wavenumber $\Delta k \sim 2\pi/a$. But (5.1) then says that the momentum of the electron is uncertain by $\Delta p \sim 2\pi\hbar/a$. So what? Well, that means the electron has on average a big kinetic energy,

$$\langle E_{\text{kin}} \rangle = \frac{1}{2m} (\Delta p)^2 \sim \frac{1}{2m} \left( \frac{2\pi\hbar}{a} \right)^2 .$$

Looks like the atom wants $a$ to be as big as possible to minimize the energy? Not so fast. We need to think about the potential energy too; the electron wants to be close to the nucleus because of the electric attraction: $U(r) = -e^2/4\pi\epsilon_0 r$. If on average the electron is a distance $a$ away from the nucleus, that means the total energy of the atom is roughly

$$\langle E_{\text{tot}} \rangle = \langle E_{\text{kin}} + E_{\text{pot}} \rangle \sim \frac{1}{2m} \left( \frac{2\pi\hbar}{a} \right)^2 - \frac{e^2}{4\pi\epsilon_0 a} . \quad (9.1)$$

Well, well. Like any system, the atom will radiate to get rid of excess energy, but the expression (9.1) is minimized for a nonzero atomic size $a_0$. The atom just won’t collapse to a smaller size because that would increase the energy!! Please work through the minimization for yourself to see how it compares to the estimate we got from purely dimensional arguments (without understanding the physics) in “Units.” Also please find the value $E_{\text{tot,min}}$ of the total energy at its minimum and compare your estimate in a previous homework. Don’t expect to get all the factors of $2\pi$ etc. to agree; that’s not the goal of dimensional analysis.

So we have proposed answers to puzzles (i–ii), the sizes of atoms and their stability. Most of the rest of the course will be dedicated to exploring the implications of the laws (I–IV). This handout has sketched how these laws take care of problems (i–iii) at the start of these notes. (We’ll discuss the others soon.)

We still have to address the more advanced question not of estimating the size of the lowest-energy state of atoms, but of calculating the sizes and energies of all the states, including the excited states. It is the excited states which radiate to create the spectral lines. Then we’ll start looking at all those phenomena of chemistry and so on.