Fermi Pressure

We saw in HW that electrons in a metal are far from a classical gas.

Now, one electron in a box of say 1 cm$^2$ exerts a tiny uncertainty pressure. Even $N$ electrons (the number of atoms in a 1 cm$^3$ sample) would exert negligible pressure if all were in the ground state. But Pauli says they can't all be in the ground state!

Idealize the chunk of metal by saying:

- The nuclei and all but 1 valence electron/nucleus form inert cores of charge $+e$.
- Each electron feels only an average charge density from these positive cores and the other electrons, which cancel. Thus each electron is free (and hence all are independent).

So the lowest-energy state comes by filling up free particle-in-a-box states, 2 electrons per allowed state.

The allowed states are characterized by 3 integers $\mathbf{n} = (n_x, n_y, n_z)$.

Each has energy $\frac{\hbar^2 \pi^2}{2mL^2} (\mathbf{n})^2$. To place $N$

valence electrons with minimal total energy, we start at $\mathbf{n} = 0$ and fill outward in spherical shells till we get to some level $\mathbf{n}$.

We'll see that $n_f$ is huge, so we can approximate the number of states with $\mathbf{n}^2 < (n_f)^2$ by the volume $\frac{4\pi}{3} (n_f)^3$.

So fix $n_f$ by requiring $N = \frac{2}{8} \frac{4\pi}{3} (n_f)^3$.

\[ n_f = \left( \frac{3N}{\pi} \right)^{1/3} \]

or $n_f = \left( \frac{3N}{\pi} \right)^{1/3}$ Yes, it's huge for a macroscopic sample.
What's the ground state energy? By separation of variables,*

\[ E = \sum_{\text{occupied}} (\text{energy of 1-particle state}) \]

\[ = \int_{10^{3/5} \eta_{K}}^{\eta_{K}} \left( \frac{2}{8} \pi \eta^{2} \Delta \eta \right) \frac{h^{2} \pi^{2}}{2mL^{2}} \eta^{2} = \pi \frac{h^{2} \pi^{2}}{2mL^{2}} \frac{1}{5}(\eta_{K})^{5} \]

\[ = \frac{h^{2} \pi^{2}}{10mL^{2}} \left( \frac{3N}{\pi} \right)^{5/3} = \frac{h^{2}}{10m} \pi^{-2/3} \pi^{4/3} \left( \frac{3N}{V} \right)^{5/3} \]

The uncertainty pressure is

\[ \frac{dE}{dV} = \frac{h^{2}}{10m} \pi^{4/3} \left( \frac{3N}{V} \right)^{5/3} \]

It's intensive - makes sense.

Why doesn't this pressure explode the metal? Normally it's balanced by the electrostatic attraction of the electrons to the nuclei.

If we upset this balance by squeezing, \((\Psi)\) is a guide to how much energy it costs.

Evaluating, we need that the density of sodium metal is

\[ \frac{N}{V} = \left( 6.45 \, \text{a}_0 \right)^{-3} \]

\[ \frac{dE}{dV} = \frac{\left( 10^{-24} \, \text{J} \, s \right)^{2}}{10 \cdot 9 \cdot 10^{-27} \, \text{kg}} \frac{2}{3} \pi^{4/3} \left( 6.45 \cdot 0.05 \, \text{nm} \right)^{-5} 3^{5/3} = 6 \cdot 10^{9} \, \text{J} \, \text{m}^{-3} \]

The measured compression modulus of metallic sodium is \(6.5 \cdot 10^{9} \, \text{J} \, \text{m}^{-3}\).

In fact most ordinary solids have moduli around \(10^{10} \, \text{J} \, \text{m}^{-3}\), so it's satisfying to understand this in terms of fundamental constants. *Our \( n_{i} \)'s are positive integers so we only want one octant of the sphere.*