Physics 503: Geometry, Relativity, and Gravitation
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> "When a god announced to the Delians through an oracle that, in order to be liberated from the plague, they would have to make an altar twice as great as the existing one, the architects were much embarrassed in trying to find out how a solid could be made twice as great as another one. They went to consult Plato, who told them that the god had not given the oracle because he needed a doubled altar, but that it had been declared to censure the Greeks for their lack of respect for geometry." — Theon of Smyrna

This course is both an introduction to the general theory of relativity as a physical theory of gravitation and a general exposition of basic differential geometric methods in physics. Ideas like tensor analysis on curved spaces, curvature, and so on are developed ab initio and applied not only to gravitation but (briefly) to other geometrical field theories as well.
Geometric methods are playing an ever-increasing role in many branches of physics, including hard and soft condensed matter theory, particle physics, and hydrodynamics.

General prerequisites:
Electromagnetism and special relativity. Many-variable calculus and linear algebra.

Required text:
S. M. Carroll, *Spacetime and Geometry*

Books on Relativity and Gravity:
M. Berry, *Principles of Cosmology and Gravitation*
R. Dicke, *The Theoretical Significance of Experimental Relativity*
R. Feynman, *Feynman Lectures on Gravitation* (Addison-Wesley, 1995)
James B. Hartle, *Gravity: An Introduction to Einstein's General Relativity*
Landau and Lifshitz, *The Classical Theory of Fields*
N.D. Mermin, *Space and Time in Special Relativity* (a classic)
Misner, Thorne, and Wheeler, *Gravitation*
Schrödinger, *Space-Time Structure* (a classic)
B. Schutz, *A First Course on General Relativity*
B.F. Schutz, *Gravity from the ground up* (Cambridge University Press, 2003)
Taylor and Wheeler, *Spacetime Physics*
Weinberg, *Gravitation and Cosmology* (Wiley, 1972)
C. Will, *Theory and Experiment in Gravitational Physics* (Cambridge, 1993)
Zhang, *Special Relativity and its Experimental Foundations* (World Scientific, 1997)

Books on Geometry:
B.A. Dubrovin, A.T. Fomenko, S.P. Novikov, *Modern geometry—methods and applications*
Flanders
T. Frankel, *The geometry of physics* (Cambridge 1997)
C. Isham, *Modern Differential Geometry for Physicists*
Nakahara
B. Schutz, *Geometrical methods of mathematical physics*
Spivak

Books on Cosmology:

Alpher and Herman, *Genesis of the Big Bang* (Oxford 2001)
Hawking and Ellis, *The large scale structure of space-time*
J. A. Peacock, *Cosmological physics* (CUP, 1999)
P.J.E. Peebles, *Principles of physical cosmology*

Books on Relativistic Astrophysics:

S. K. Chakrabarti, ed. *Observational Evidence for Black Holes in the Universe* (Kluwer, 1999)
Chandrasekhar, *Mathematical theory of black holes*
N. Straumann, *General Relativity and Relativistic Astrophysics*

Miscellaneous books:

R. Aris, *Vectors, Tensors, and the Basic Equations of Fluid Mechanics*
Michael D. Lemonick, *Echo of the Big Bang*
A. Pais, *Subtle is the Lord* (historical)
R. Shankar, *Basic Training in Mathematics* (Plenum, 1995)
John Archibald Wheeler, *A journey into Gravity and Spacetime* (W. H. Freeman and Co., 1990)

Outline

[optional sections in brackets]

0. What is physics anyway?

   "The image of the theorist that emerges is of a surprisingly unsophisticated individual who must anthropomorphize nature to understand it: I like symmetry and beauty; ergo, nature likes symmetry and beauty. It reminds one of the Parisian animal trainer who teaches his bear to respond to voice commands and concludes that bears speak French." — Dick Teresi

   - Why relativity usually makes you feel cheated.

1. Prolog: On geometry
   - Metric properties are flat to order r^2.
   - Geodesics.
   - Curvature as bending; principal curvatures.
   - Curvature as error in circumference, area, angles.
   - Some curvature is intrinsic.
   - Geodesics.
   - The pseudosphere: negative curvature, defocusing geodesics.

2. Prolog, part 2: On relativity

   "Time is nature's way of making sure things don't all happen at once." — Woody Allen

   - Universality of c contradicts Galilean invariance. Æther theory and its discontents.
   - Maxwell frames; affine-conformal group; Einstein frames; Poincaré group. A digression into an alternate theory.
   - Spacetime diagrams. Geometric characterization of Einstein frames. How to boost to a rest frame without changing c.
   - Before/simultaneous/after is replaced by timelike/lightlike/spacelike.
   - Time dilation. An apparent paradox. Unaccelerated motion maximizes the proper time in SR. P of E suggests that free fall with gravity is governed by the geodesic equation.
   - Lorentz group and its Lie algebra.
   - Foundational questions. Mach's principle.

3. Prolog, part 3: On gravitation

   "The axiomatic method has many advantages, which are similar to the advantages of theft over honest work." — B. Russell

- Physical postulates vs. operational definitions vs merely mathematical facts. Æsthetics versus dynamics.
- Basic operational concepts. Falling frames.
- Newtonian coordinate frames. Galilean transformations. Newton's main hypothesis.
- Symmetry in physics. Active vs passive transformations.
- How to flush hidden geometrical objects out into the open. Preview of our strategy.
- Universality of gravitation; Eötvös experiment. The hypothesis that metric carries gravity.
- Principle of Equivalence. Metric geometry is particularly adapted to give it. Gravitational red shift.
- Metric seems to contain far too many degrees of freedom.

4. Geometry on manifolds

> "I cannot abstain to play the role of an (often unwelcome) intermediary in this drama between mathematics and physics, who fertilize each other in the dark, and deny and misconstrue one another face to face. — H. Weyl

   - Tangent vectors: coordinate-invariant constructions.
   - Metric; concept of tensor; coordinate transformations.
   - Metrics are locally flat in n dimensions.
   - Example: momentum 4-vector and its conservation.
   - Lie bracket. Covectors, contraction, adjoints, gradient; transformation laws. Examples.

5. Electrodynamics on curved spacetime
   - Lorentz law defines E, B in any inertial frame.
   - Deduce the transformation of E, B and get intrinsic tensor F.
   - Maxwell's homogeneous equations are fully covariant.
   - Differential forms. Exterior derivative.
   - Maxwell's inhomogeneous equations; Hodge star; tensor densities.
   - [Poincaré lemma.]
   - Gauge invariance $\leftrightarrow$ automatic conservation.

6. Connections
   - Free fall; naive derivative of a vector in flat space; curvilinear coordinates and Γ.
   - Coordinate-invariant definition of connection.
   - Torsion and metricity conditions.
   - The Riemann=Levi-Civita=Christoffel connection.
   - Parallel transport in general; recover our simple 2d definition; the geodesic equation.
   - Light rays and free fall are described by geodesics; Newtonian limit.

7. Curvature

> "'What's the good of Mercator's North Poles and Equators
> Tropics, Zones, and Meridian Lines?'
> So the Bellman would cry: and the crew would reply
> 'They are merely conventional signs!'"
> — *The Hunting of the Snark*

   - R as $[\nabla, \nabla]$
   - Curvature as error of transport
   - Symmetries of R tensor; counting its independent components; contractions of R
   - Covariant exterior derivative. Bianchi identity.
   - Geodesic deviation. The Einstein tensor; automatic conservation.

8. More about R
   - Lorentz-valued 2-form.
   - $\binom{p}{q}$-valued n-forms; the D operator.
   - [Tetrads; algorithm to compute R.]
   - Meaning of the Einstein tensor. The Einstein field equations.
   - Summary of mathematical half of this course

9. Stress
   - Fluids. Perfect fluids. Dust.
   - Number 4-current and stress tensor for dust.
   - [Energy density; temperature, entropy.]
   - Stress tensor in all generality; symmetry, conservation.

10. The weak-field limit
    - General form; geometrical units.
    - Newtonian limit; Lorentz gauge.

11. Gravitational Radiation

    "When a young man in my laboratory uses the word 'universe' I tell him it is time to leave." — Ernest Rutherford

    - Linearized equations; two polarizations.
    - Detection.
    - Generation; binary pulsar; quadrupole approximation.

12. Energy of Gravity
    - $T,\ t,\ \tau$; τ is conserved.
    - Calculation of t; $\langle t_{\beta\nu}\rangle = \frac{1}{16\pi}k_\beta k_\nu(|A_+|^2 + |A_-|^2)$.
    - Binary pulsar loss. 1993 Nobel Prize.

13. Spherical symmetry and the classic tests of GR
    - General static isotropic metric.
    - Schwarzschild.

14. More about Schwarzschild
    - Kepler problem.
    - Perihelion shift.
    - Light bending.

15. Spin and precession
    - Spin in SR.
    - Spin along geodesics; geodetic precession.
    - [Fermi-Walker transport; Thomas precession.]

16. Stellar structure
    - Oppenheimer-Volkov equation.
    - Schwarzschild's exact solution.
    - Chandrasekhar limit.
    - Neutron stars.

17. Gravitational collapse

    "Voilà les géometres qui obligent un homme malgré lui d'être persuadé et le convainquent avec tyrannie." — Montesquieu

    - Spherically symmetric time-dependent metric; Birkhoff theorem
    - Comoving coordinates
    - Collapse of homogeneous fluid
    - Tying on the exterior solution
    - Black holes

18. The Universe

    "We can assert with certitude that the universe is all center, or that the center of the universe is everywhere and the circumference nowhere." — Giordano Bruno, 1584

    - Scales; homogeneity and isotropy
    - Constant-curvature 3-metrics
    - The Big Bang

19. [What is supergravity?]

    "Above stands the marble smile of implacable Nature, which has endowed us more with longing than with intellectual capacity." — Einstein, on unified field theory

20. [Yang-Mills theory]
    - Vector bundles
    - Connections and curvature as $[\nabla, \nabla]$
    - Gauge-transformation properties

21. [Yang-Mills bis]
    - Getting rid of matter: principal bundles
    - Trivial, frame, and G/H bundles
    - SO(3)/SO(2) and Ehresman connections
    - Holonomy
    - First and second Chern classes; instantons

22. Valedictory