Quantum Fluctuations in the Frustrated Antiferromagnet

$\text{Sr}_2\text{Cu}_3\text{O}_4\text{Cl}_2$

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Abstract

Sr$_2$Cu$_3$O$_4$Cl$_2$ is an antiferromagnet consisting of weakly coupled CuO planes which comprise two weakly interacting antiferromagnetic subsystems, I and II, which order at respective temperatures $T_I \approx 390$K and $T_{II} \approx 40$K. Except asymptotically near the ordering temperature, these systems are good representations of the two-dimensional quantum spin 1/2 Heisenberg model. For $T < T_{II}$ there are four low-energy modes at zero wave vector, three of whose energies are dominated by quantum fluctuations. For $T_{II} < T < T_I$ there are two low energy modes. The mode with lower energy is dominated by quantum fluctuations.
Our calculations of the energies of these modes (including dispersion for wave
vectors perpendicular to the CuO planes) agree extremely well with the experi-
mental results of inelastic neutron scattering (in the accompanying paper) and
for modes in the sub meV range observed by electron spin resonance. The pa-
rameters needed to describe quantum fluctuations are either calculated here or
are taken from the literature. These results show that we have a reasonable
qualitative understanding of the band structure of the lamellar cuprates needed
to calculate the anisotropic exchange constants used here.
I. INTRODUCTION

There has been a resurgence of interest in low-dimensional magnetism due in part to the desire to understand high-$T_c$ superconductivity. The lamellar copper oxide systems, when suitably doped give rise to a family of superconductors with $T_c$’s in the range about 30K.\(^1\) In these systems the Cu ions are essentially in a 3d\(^9\) configuration. Due to a large on-site Coulomb interaction, the states of this system which are accessible at ambient temperature have one hole per Cu ion, and hence the manifold of such accessible states is described by a spin 1/2 Hamiltonian having antiferromagnetic interactions, which are strongest between nearest neighboring Cu ions in the CuO\(_2\) plane. That this system is a nearly perfect realization of the two dimensional (2D) spin 1/2 quantum Heisenberg model has been established by a wide variety of experiments.\(^2\)

Recently, a variant of this system, Sr\(_2\)Cu\(_3\)O\(_4\)Cl\(_2\) (2342), has been shown to display very interesting magnetic properties.\(^3\)–\(^5\) The structure of this system\(^6\) is one in which an additional Cu ion (which we refer to as a CuII ion) is inserted at the center of alternate Cu plaquettes of the usual copper lattice, whose ions we refer to as CuI’s. Although all the Cu ions are chemically equivalent, they play very different roles insofar as magnetism is concerned. The CuI’s order at a relatively high temperature ($T_I = 386$K) and have properties similar to those of other lamellar cuprate antiferromagnets.\(^2\) With respect to the isotropic exchange interactions, the coupling between CuI and CuII ions is frustrated. As a result, the CuII’s order independently at a much lower temperature, $T_{II} = 39.6$K into the magnetic structure shown in Fig. 1. For $T_{II} < T < T_I$ a very small residual anisotropic exchange interaction causes the CuII spins to have a small ferromagnetic moment, the study of which\(^4\) led to the determination of the magnetic structure which has recently been confirmed by neutron diffraction.\(^7\) The study of the statics also led to the determination of several coupling constants in the Hamiltonian used to model this system.

A natural continuation of this study was to investigate the dynamics of this system, and in the accompanying paper\(^7\) (which we refer to as I) an inelastic neutron scattering study of this system is reported. One interesting result of these experiments was that although the coupling between the CuI’s and CuII’s is frustrated in the mean-field sense, the spin-wave spectrum showed an incontrovertible signature of interactions between these subsystems.\(^5,7\) The nature
of this coupling was described by Shender in a seminal paper. Although this phenomenon has been identified in other materials, the effect of this coupling, caused by quantum fluctuations, is perhaps the most dramatic in the system 2342, as described briefly previously and in more detail in I. As the CuII system orders for $T < T_{II}$, the small gap spin-wave energies are found to increase sharply. This increase indicates that even though the CuI-CuII coupling is frustrated in the mean-field sense, quantum fluctuations lead to a significant interaction between sublattices. A less obvious type of frustration arises with respect to the in-plane anisotropy associated with the bond anisotropy of the exchange interactions. When the moments lies in the easy plane, the exchange tensor for spins $i$ and $j$ in the plane has different values for directions parallel and perpendicular to the $i$-$j$ bond. However, within mean field theory this anisotropy disappears when the average over all bonds is taken. But as before, there is a significant residual interaction due to quantum fluctuations which gives rise to in-plane anisotropy. Finally, even classically frustration can be removed by exchange anisotropy which has a form similar to the dipolar interaction. We will refer to such exchange anisotropy as pseudo-dipolar.

The purpose of the present paper is to calculate the spin-wave spectrum in order to give a theoretical interpretation to the data presented in I. From the discussion so far it is clear that most of these phenomena are outside the scope of linearized spin-wave theory. What is required is a nonlinear spin-wave analysis, i.e. an analysis which includes the effects of quantum fluctuations. In fact, from an analysis of the magnetic structure of the cuprates it was shown that there are several perturbations away from the linear analysis of the isotropic Heisenberg model that one must consider. These are the ones mentioned above, namely, a) quantum fluctuations of otherwise frustrated interactions, b) quantum fluctuations of the anisotropic in-plane exchange interactions, and c) pseudo-dipolar exchange anisotropy between the CuI and CuII subsystems. In a simplified way, one can categorize these effects in the way they contribute to the spin-wave energies, which is given by the famous formula

\[ \omega = \sqrt{2H_E H_A} , \]  

where $H_E$ ($H_A$) is the exchange (anisotropy) field and we work in units such that $\omega$, $H_E$, and $H_A$ are all energies, usually given in meV ($1\text{meV}/k_B = 11.6K, 1\text{meV}/\hbar = 241.8\text{Ghz}$.) We will see that the out-of-plane anisotropy of the exchange interactions gives rise to a corresponding
out-of-plane anisotropy field $H_A^{\text{out}}$ which has been understood in terms of the out-of-plane anisotropy in the exchange interactions without reference to fluctuations.\textsuperscript{12,13} In contrast, the in-plane anisotropy of the exchange interactions, when summed over bonds, averages to zero and therefore only contributes when fluctuations are taken into account.\textsuperscript{12,13} The mechanism studied by Shender\textsuperscript{8} contributes to $H_A$ except for the Goldstone mode, whose energy becomes nonzero only when lattice anisotropy is introduced.

One might expect that the number of coupling constants might be so large that no useful information or test of the theory would be possible. As it happens, the fit to the energy of the gaps is overdetermined and the agreement between theory and experiment in some instances is quite remarkable, as can be seen in I. The observation of the modes whose energy depends on the in-plane anisotropy leads to the determination of the in-plane anisotropy of the exchange interactions. These quantities are difficult to obtain experimentally. Their values can be compared to calculations\textsuperscript{12-14} based on the electronic structure of the cuprates the knowledge of which may lead to a better understanding of the high-$T_c$ superconductors.

One should recognize that at the moment inelastic neutron scattering does not easily detect modes in the sub meV range of energy. As a result neutron scattering experiments have not detected those in-plane modes whose energy depends only on the in-plane anisotropy. Recently, however, the modes in the sub meV range of energy have been observed by ESR experiments of the group at RIKEN.\textsuperscript{15,16} The mere existence of these modes tends to confirm the spin-wave calculations. Moreover, the fact that they are found in the predicted range of energy strongly supports the theoretical calculations in this paper.

Briefly, this paper is organized as follows. In Sec. II the Hamiltonian with its various anisotropic exchange interactions is specified. In Sec. III we start by discussing briefly the framework within which the calculations are to be done and we give the Dyson–Maleev transformation\textsuperscript{17} to boson operators. In Sec. IV the isotropic exchange Hamiltonian is discussed, first within harmonic theory and then including spin-wave interactions, which are essential to obtain a qualitatively correct spectrum. In Sec. V the various anisotropies are included in an effective quadratic spin-wave Hamiltonian. In Sec. VI we give explicit results for the spin-wave energies for the case when the transverse wave vector is zero and show the
comparison of our calculations with the recent experiments of the MIT group. In Sec. VII intensities of modes are discussed, with numerical results given for zero wave vector relative to the Bragg peaks for CuI and CuII. Our conclusions are summarized in Sec. VIII.
II. HAMILTONIAN

The Hamiltonian that we intend to treat is written as

$$\mathcal{H} = \mathcal{H}_1 + \mathcal{H}_2,$$

where $\mathcal{H}_1$ includes almost all the significant interactions, namely all the intraplanar interactions and the unfrustrated interactions between nearest neighbors in adjacent CuO planes and $\mathcal{H}_2$ includes small residual anisotropic interplanar interactions involving CuII spins. Since this latter term is totally negligible except for extremely small wave vector and for the lowest energy mode, it is only necessary to include contributions from $\mathcal{H}_2$ evaluated at zero wave vector. Since the effects of $\mathcal{H}_2$ are only relevant to the extremely low frequency spectrum, we defer consideration of $\mathcal{H}_2$ until Secs. V.4 and V.5.

Thus we write $\mathcal{H}_1$ in tensor notation as

$$\mathcal{H}_1 = \frac{1}{2} \sum_{\langle i,j \in \mathcal{I} \rangle} S_i J_{ij} S_j + \sum_{\langle i \in \mathcal{I}, j \in \mathcal{II} \rangle} S_i J_{I-II} S_j + \sum_{\langle j, j \in \mathcal{II} \rangle} S_i J_{II} S_j + \sum_{i \in \mathcal{I}} J_3 S_i \cdot S_{i+\frac{1}{2}c^\pm},$$

where $i \in \mathcal{I}(i \in \mathcal{II})$ means that site $i$ runs over CuI (CuII) sites and $\langle \rangle$ restricts the summation to nearest neighbors of the indicated type in the same Cu-O plane. The only unfrustrated coupling between planes is that ($J_3$) between CuI's directly above or below one another. We will allow the couplings $J_I, J_{I-II},$ and $J_{II}$ to be anisotropic, whereas for simplicity we take $J_3$ to be isotropic. Here and below we use a hybrid notation for site labels in which the label $i + \mathbf{r}$ indicates a site at position $\mathbf{r}$ with respect to site $i$. In $\mathcal{H}_2$ we include the interplanar CuI - CuII and CuII-CuII couplings whose isotropic parts are frustrated.

We first discuss the principal axes of the exchange tensor $J_I$ associated with a bond between nearest neighboring CuI spins in a CuO plane. This bond is invariant with respect to two mirror planes: one in the CuO plane and the other perpendicularly bisecting the CuI - CuI bond in question. Accordingly, the principal axes of the CuI - CuI exchange tensor between nearest neighbors lie along the three crystal $(1,0,0)$ directions, just as they would be in the absence of the CuII's. In that case, the exchange tensor will have different values corresponding to the directions i) along the bond in question, ii) perpendicular to the bond in question but in the CuO plane, and iii) along the crystal $c$ direction. The principal axes of the other in-plane
interactions are similarly fixed by symmetry.\(^\text{13,18}\) Then the Hamiltonian \(\mathcal{H}_1\) may be written as follows

\[
\mathcal{H}_1 = \frac{1}{2} \sum_{i \in I} \sum_{\delta_{1}} \left( J_{I} J_{I}^{\perp} \mathbf{S}_i \cdot \mathbf{S}_{i+\delta_{1}} + J_{I}^{\parallel} \mathbf{S}_i \cdot \hat{\mathbf{e}}_{1} \right) + \sum_{i \in I} \sum_{\delta_{2}} \left( J_{II}^{\parallel} \mathbf{S}_i \cdot \hat{\mathbf{e}}_{2} \right) + J_{J} \mathbf{S}_i \cdot \frac{1}{2} \mathbf{e}_{z},
\]

where \(\delta_{1}, \delta_{2}\) labels the nearest neighbor vectors in the plane connecting adjacent CuI's (CuII's) and \(\delta_{1}, \delta_{2}\) labels vectors in the CuO plane which give the displacements of nearest neighboring CuI's relative to a CuII, and hat indicates a unit vector. Also \(\mathbf{e}_{1}, \mathbf{e}_{2}\) are unit vectors in the CuO plane which are perpendicular to, respectively, \(\delta_{1}, \delta_{2}, \) and \(\delta_{2}\).

We separate the Hamiltonian \(\mathcal{H}_1\) into an isotropic part, \(\mathcal{H}_0\), and an anisotropic perturbation, \(\mathcal{H}'\). For that purpose we write

\[
\Delta J_1 = \frac{1}{2} (J_I + J_{I}^{\perp}) - J_I^{\parallel}, \quad \Delta J_{12} = \frac{1}{2} (J_{I}^{\parallel} - J_{I}^{\perp}), \quad J_{J}^{\perp} = \frac{1}{2} (J_{I}^{\parallel} + J_{II}^{\parallel}) - J_{J}^{\parallel},
\]

\[
\delta J_1 = \frac{1}{2} (J_I^{\parallel} - J_I^{\perp}), \quad \delta J_{12} = \frac{1}{2} (J_{I}^{\parallel} - J_{II}^{\parallel}) - J_{II}^{\parallel}, \quad \delta J_{I} = \frac{1}{2} (J_{I}^{\parallel} - J_{II}^{\parallel}),
\]

\[
\mathcal{J} = \frac{1}{3} (J_I + J_{I}^{\perp}) + J_I^{\parallel}, \quad \mathcal{J}_{12} = \frac{1}{3} (J_{I}^{\parallel} + J_{I}^{\perp} + J_{II}^{\parallel}) + \mathcal{J}_{II},
\]

Thus the \(\Delta J_1\)'s describe the out-of-plane anisotropy (i.e. the energy which gives rise to an easy-plane) which is responsible for the 5 meV anisotropy gap in the spin-wave spectra of cuprates which do not have CuII's. Similarly, the \(\delta J_1\)'s describe the in-plane anisotropy (i.e. the anisotropy within the easy plane) and they i) are responsible for the weak ferromagnetic moment\(^3\)\(^4\) induced in the CuII subsystem by the staggered moment in the CuI subsystem and ii) contribute to the macroscopic or phenomenological four-fold anisotropy constant \(K_4\).\(^3\)\(^4\)

(We shall see later that \(\mathcal{H}_2\) also contributes to \(K_4\).) Note that \(\delta J_{12}\) is what was called \(J_{pl}\) in Refs. 3 and 4, but differs by a factor of two from its definition in Refs. 13 and 14. The largest coupling is \(J (J_{2}/J \approx J_{12}/J \approx 0.1\) and \(J_{3}/J \approx 10^{-3}\), while the relative anisotropies, \(\Delta J/J\) and \(\delta J/J\) are at most \(10^{-3}\).\(^3\)\(^4\)\(^13\)\(^14\)

With these notations the isotropic Hamiltonian is

\[
\mathcal{H}_0 = \frac{1}{2} \sum_{i \in I} \sum_{\delta_{1}} \mathbf{J} \mathbf{S}_i \cdot \mathbf{S}_{i+\delta_{1}} + \sum_{i \in I} \sum_{\delta_{2}} \mathbf{J} \mathbf{S}_i \cdot \mathbf{S}_{i+\delta_{2}} + J_{J} \mathbf{S}_i \cdot \mathbf{S}_{i+\frac{1}{2} \mathbf{e}_{z}},
\]

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and the anisotropic perturbation is

\[
\mathcal{H}' = -\frac{1}{2} \Delta J_1 \sum_{i \in I, \delta_1} S_i^z S_{i+\delta_1}^z - \Delta J_{12} \sum_{i \in I, \delta_{2,1}} S_i^z S_{i+\delta_{2,1}}^z - \frac{1}{2} \Delta J_2 \sum_{i \in I, \delta_2} S_i^z S_{i+\delta_2}^z \\
+ \frac{1}{2} \delta_1 J_1 \sum_{i \in I, \delta_+} \left( S_i^x S_{i+\delta_+}^x + S_i^y S_{i+\delta_+}^y \right) - \frac{1}{2} \delta J_1 \sum_{i \in I, \delta_-} \left( S_i^x S_{i+\delta_-}^x + S_i^y S_{i+\delta_-}^y \right) \\
+ \delta J_{12} \sum_{i \in I, \delta_2} \left( S_i^x S_{i+\delta_2}^x - S_i^y S_{i+\delta_2}^y \right) + \frac{1}{2} \delta J_2 \sum_{i \in I, \delta_2} \left( S_i^y S_{i+\delta_2}^y - S_i^x S_{i+\delta_2}^x \right) \\
+ \frac{1}{2} \delta J_2 \sum_{i \in I, \delta_2, j = i+\delta_2} \left[ S_i \cdot \hat{\delta}_2 | S_j \cdot \hat{\delta}_2 | - | S_i \cdot \hat{\delta}_2 | S_j \cdot \hat{\delta}_2 | \right], \tag{9}
\]

where we introduce the following sums over the \( \delta \)’s:

\[
\delta_x = \pm \frac{1}{2} a \hat{x} , \quad \delta_y = \pm \frac{1}{2} a \hat{y} , \quad \delta_+ = \pm \frac{1}{2} a (\hat{x} + \hat{y}) , \quad \delta_- = \pm \frac{1}{2} a (\hat{x} - \hat{y}) , \tag{10}
\]
as shown in Fig. 2. In Eq. (8), \( \tilde{J} = J + \frac{1}{3} \Delta J \) and similarly for the other \( J \)’s. Since the anisotropy in the \( J \)’s is so small (at most of order \( 10^{-3} \)), we henceforth drop the tildes.

It is convenient to express the spin components in a coordinate system in which one axis (the \( \xi \) axis) lies along the line of the staggered magnetization. Thus we introduce the axes \( \xi \) and \( \eta \) which are obtained from \( x \) and \( y \) by a rotation about the \( z \) axis of \( \pi/4 \). Then

\[
S^x = (S^\xi - S^\eta)/\sqrt{2} , \quad S^y = (S^\xi + S^\eta)/\sqrt{2} , \tag{11}
\]
so that

\[
\mathcal{H}' = -\frac{1}{2} \Delta J_1 \sum_{i \in I, \delta_1} S_i^z S_{i+\delta_1}^z - \Delta J_{12} \sum_{i \in I, \delta_{2,1}} S_i^z S_{i+\delta_{2,1}}^z - \frac{1}{2} \Delta J_2 \sum_{i \in I, \delta_2} S_i^z S_{i+\delta_2}^z \\
+ \frac{1}{2} \delta_1 J_1 \sum_{i \in I, \delta_+} \left( S_i^\xi S_{i+\delta_+}^\xi - S_i^\eta S_{i+\delta_+}^\eta \right) + \frac{1}{2} \delta J_1 \sum_{i \in I, \delta_-} \left( S_i^\eta S_{i+\delta_-}^\eta - S_i^\xi S_{i+\delta_-}^\xi \right) \\
- \delta J_{12} \sum_{i \in I, \delta_2} \left( S_i^\xi S_{i+\delta_2}^\xi + S_i^\eta S_{i+\delta_2}^\eta \right) + \frac{1}{2} \delta J_2 \sum_{i \in I, \delta_2} \left( S_i^\xi S_{i+\delta_2}^\xi + S_i^\eta S_{i+\delta_2}^\eta \right) \\
- \delta J_2 \sum_{i \in I, \delta_2, j = i+2\delta_2} \left( S_i^\xi S_j^\xi + S_i^\eta S_j^\eta \right) + \delta J_2 \sum_{i \in I, \delta_2} \sum_{j = i+2\delta_2} \left( S_i^\xi S_j^\xi + S_i^\eta S_j^\eta \right), \tag{12}
\]

where, in the last line, \( i \in e \) indicates that the sum is taken over only half the CuII spins, i. e. those on the \( e \) sublattice (see Fig. 1).
III. BOSON HAMILTONIAN

A. Overview of the Calculation

Since the CuI - CuII interaction is frustrated, the CuI and CuII sublattices are decoupled within mean-field theory or within harmonic spin-wave theory at zero wave vector. In other words, to calculate the energy gaps at zero wave vector we will need to include fluctuations, as first indicated by Shender. Here, in view of the myriad of terms in the Hamiltonian, we need to proceed in as systematic a way as possible. In the original work of Shender it was found that the effective coupling between sublattices, which depends on fluctuations beyond mean-field theory or beyond harmonic spin-wave theory involved energies of relative order $1/S$ with respect to energies encountered in mean-field theory. Accordingly, here we will calculate all relevant effects in the spin-wave spectrum due to anharmonic perturbations up to first order in $1/S$. Therefore we analyze perturbative contributions at one-loop order. To be more specific, we will introduce the usual Dyson-Maleev boson representation of spin operators, in terms of which anharmonic perturbations involving three (four) boson operators are of relative order $1/\sqrt{S}$ ($1/S$). This means that we treat four-operator perturbations within first-order perturbation theory and three-operator perturbations within second-order perturbation theory. In technical language, this would be done by keeping all such contributions to the wave vector and energy-dependent self-energy. Since we work to low order, a more naive approach (which is entirely equivalent to calculating the self-energy) is both convenient and easy to follow. In this naive approach one truncates all four operator terms by contracting out pairs of operators in all possible ways. This reproduces exactly the results of the one-loop diagrams obtained by treating the four operator vertices in first order perturbation theory. In addition, we would note that all nonHermitian terms at order $1/S$ do not contribute to first order energies. So, at order $1/S$ we simply discard nonHermitian terms. Since the three-operator terms are of interest in producing small gaps, we will follow a calculational method which is strictly correct only at zero wave vector. The fact that our treatment the small perturbations have the wrong dependence on wave vector is irrelevant because their effect is only nonnegligible very near zero wave vector. To avoid the algebraic complexities due to the fact that the magnetic structure has six sublattices,
we simply construct, by the methods mentioned above, the effective quadratic Hamiltonian which includes all the self-energy corrections at order 1/$S$. As a check that our calculations are really as consistent as we claim, we verify that the gaps have the expected dependence on the perturbations. In other words, when the perturbations are known to not produce gaps, our calculations reproduce that result. This type of check indicates that, for instance, our treatment of three-operator terms in second order perturbation theory is consistent with our treatment of four-operator terms in first order perturbation theory.

**B. Transformation to Bosons**

We make the following Dyson-Maleev transformation\textsuperscript{17} to bosons ($a$, $b$, \ldots $f$):

\[
S_a^+ = \sqrt{2S}a^\dagger, \quad S_a^- = \sqrt{2S}a, \quad S_a^\xi = S - a^\dagger a
\]

\[
S_b^+ = \sqrt{2S}b^\dagger, \quad S_b^- = \sqrt{2S}b, \quad S_b^\xi = S - b^\dagger b
\]

\[
S_c^+ = \sqrt{2S}c^\dagger, \quad S_c^- = \sqrt{2S}c, \quad S_c^\xi = S - c^\dagger c
\]

\[
S_d^+ = \sqrt{2S}d^\dagger, \quad S_d^- = \sqrt{2S}d, \quad S_d^\xi = S - d^\dagger d
\]

\[
S_e^+ = \sqrt{2S}e^\dagger, \quad S_e^- = \sqrt{2S}e, \quad S_e^\xi = S - e^\dagger e
\]

\[
S_f^+ = \sqrt{2S}f^\dagger, \quad S_f^- = \sqrt{2S}f, \quad S_f^\xi = S - f^\dagger f
\]

where $S^\pm = S^u \pm iS^z$, $\phi(x) = 1 - x^\dagger x/(2S)$, and we have left the site labels implicit. In bosonic variables the isotropic interaction between spins assumes the form

\[
S_{ai} \cdot S_{bj} = S \left( a_i^\dagger a_i + b_j^\dagger b_j + a_i b_j + a_j^\dagger b_j^\dagger \right) - \frac{1}{2} \left( b_j^\dagger b_j a_i + b_j^\dagger a_i^\dagger a_i + 2a_i^\dagger a_i b_j^\dagger b_j \right)
\]

\[
S_{ai} \cdot S_{ej} = S \left( a_i^\dagger a_i + e_j^\dagger e_j + a_i e_j + a_j^\dagger e_j^\dagger \right) - \frac{1}{2} \left( e_j^\dagger e_j a_i + e_j^\dagger a_i^\dagger a_i + 2a_i^\dagger a_i e_j^\dagger e_j \right)
\]

\[
S_{ai} \cdot S_{fj} = S \left( -a_i^\dagger a_i - f_j^\dagger f_j + a_i^\dagger f_j + f_j^\dagger a_i \right) - \frac{1}{2} \left( a_i^\dagger f_j^\dagger f_j^\dagger f_j + f_j^\dagger a_i^\dagger a_i - 2a_i^\dagger a_i f_j^\dagger f_j \right)
\]

\[
S_{bi} \cdot S_{ej} = S \left( -b_i^\dagger b_i - e_j^\dagger e_j + b_i^\dagger e_j + e_j^\dagger b_i \right) - \frac{1}{2} \left( b_i^\dagger e_j^\dagger e_j e_j + e_j^\dagger b_i^\dagger b_i - 2b_i^\dagger b_i e_j^\dagger e_j \right)
\]

\[
S_{bi} \cdot S_{fj} = S \left( b_i^\dagger b_i + f_j^\dagger f_j + b_i^\dagger f_j + f_j^\dagger b_i \right) - \frac{1}{2} \left( b_i^\dagger f_j^\dagger f_j^\dagger f_j + f_j^\dagger b_i^\dagger b_i + 2b_i^\dagger b_i f_j^\dagger f_j \right)
\]

\[
S_{ei} \cdot S_{fj} = S \left( e_i^\dagger e_i + f_j^\dagger f_j + e_i^\dagger f_j + e_i f_j \right) - \frac{1}{2} \left( e_i^\dagger f_j^\dagger f_j^\dagger f_j + f_j^\dagger e_i^\dagger e_i + 2e_i^\dagger e_i f_j^\dagger f_j \right)
\]
The other interactions can be obtained by appropriate relabeling of boson variables.

The effective bilinear spin-wave Hamiltonian is of the form (see below)

$$\mathcal{H} = \sum_q \left[ A(q)_{\mu \nu} \xi_\mu(q)^\dagger \xi_\nu(q) + \frac{1}{2} B(q)_{\mu \nu} \xi_\mu(q)\xi_\nu(-q) + \frac{1}{2} B(q)_{\mu \nu} \xi_\mu(q)^\dagger \xi_\nu(-q) \right],$$

(15)

where $\xi_\mu(q) = a(q)$ and so forth (in order $b$, $c$, $d$, $e$, and $f$). Here

$$\xi_\mu^\dagger(i) = \frac{1}{\sqrt{N_{uc}}} \sum_q e^{iq\cdot r_i} \xi_\mu(q),$$

(16)

where $N_{uc}$ is the number of unit cells.

C. Spin-wave Spectrum. General Considerations

The transformation to normal mode operators $\tau_k(q)$ is

$$\xi_i^\dagger(q) = \sum_j P_{ij}(q)^* \tau_j^\dagger(q) + \sum_j Q_{ij}(q)^* \tau_j(-q)$$

$$\xi_i(-q) = \sum_j Q_{ij}(q)^* \tau_j^\dagger(q) + \sum_j P_{ij}(q)^* \tau_j(-q).$$

(17)

To preserve the commutation relations we require that

$$P(q)P^\dagger(q) - Q(q)Q^\dagger(q) = \mathcal{I}, \quad P(q)Q^\dagger(q) - Q(q)P^\dagger(q) = 0,$$

(18)

where $\mathcal{I}$ is the unit matrix.

The transformation inverse to Eq. (17) is therefore

$$\tau_j^\dagger(q) = \sum_k P_{kj}(q)\xi_k^\dagger(q) - \sum_k Q_{kj}(q)\xi_k(-q)$$

$$\tau_j(q) = -\sum_k Q_{kj}(q)^* \xi_k^\dagger(-q) + \sum_k P_{kj}(q)^* \xi_k(q).$$

(19)

The equation that determines the normal modes is

$$[\tau_j(q), \mathcal{H}]_\pm = \omega_j(q)\tau_j(q)$$

(20)

which gives

$$\begin{bmatrix} A(q) & B(q) \\ -B(q) & -A(q) \end{bmatrix} \begin{bmatrix} P_j(q) \\ Q_j(q) \end{bmatrix} = \omega_j(q) \begin{bmatrix} P_j(q) \\ Q_j(q) \end{bmatrix}.$$
where \( \mathbf{P}_j \) is the column vector with components \( P_{1j}, P_{2j}, \ldots, P_{nj} \) and \( \mathbf{P} = [\mathbf{P}_1, \mathbf{P}_2, \ldots, \mathbf{P}_n] \) and similarly for the \( \mathbf{Q}' \)s.

From now on the arguments are always \( \mathbf{q} \). Then

\[
[A + B][\mathbf{P}_j + \mathbf{Q}_j] = \omega_j[\mathbf{P}_j - \mathbf{Q}_j]
\]

\[
[A - B][\mathbf{P}_j - \mathbf{Q}_j] = \omega_j[\mathbf{P}_j + \mathbf{Q}_j].
\]

Therefore

\[
[A + B][A - B][\mathbf{P}_j - \mathbf{Q}_j] = \omega_j^2[\mathbf{P}_j - \mathbf{Q}_j].
\]

Hence, the squares of the spin-wave energies are the eigenvalues of the matrix

\[
\mathbf{D}(\mathbf{q}) \equiv [\mathbf{A}(\mathbf{q}) + \mathbf{B}(\mathbf{q})] \times [\mathbf{A}(\mathbf{q}) - \mathbf{B}(\mathbf{q})].
\]

Roughly speaking the matrices \( \mathbf{A} + \mathbf{B} \) and \( \mathbf{A} - \mathbf{B} \) reproduce the stiffnesses in the two directions transverse to the sublattice magnetization.

As we shall see later, for the Hamiltonian of the form of \( \mathcal{H}_1 \) these dynamical matrices assume the form

\[
\mathbf{A}(\mathbf{q}) =
\begin{pmatrix}
  a_{11} & a_{12}c_+ & a_{12}c_- & 0 & a_{15}e_x & a_{16}e_x^* \\
  a_{12}c_+ & a_{11} & 0 & a_{12}c_- & a_{16}e_y & a_{15}e_y \\
  a_{12}c_- & 0 & a_{11} & a_{12}c_+ & a_{16}e_y^* & a_{15}e_y^* \\
  0 & a_{12}c_- & a_{12}c_+ & a_{11} & a_{15}e_x & a_{16}e_x^* \\
  a_{15}e_x^* & a_{16}e_y & a_{16}e_y^* & a_{15}e_x & a_{55} & a_{56} \frac{c_x + c_y}{2} \\
  a_{16}e_x & a_{15}e_y^* & a_{15}e_y & a_{16}e_x^* & a_{56} \frac{c_x + c_y}{2} & a_{55}
\end{pmatrix}
\]

\( (25a) \)

and

\[
\mathbf{B}(\mathbf{q}) =
\begin{pmatrix}
  b_{11} & (b_{12}c_+ + 2J_3S) & (b_{12}c_-) & 0 & b_{15}e_x & b_{16}e_x^* \\
  (b_{12}c_+ + 2J_3S) & b_{11} & 0 & b_{12}c_- & b_{16}e_y & b_{15}e_y \\
  (b_{12}c_-) & 0 & b_{11} & (b_{12}c_+ + 2J_3S) & b_{15}e_x & b_{16}e_x^* \\
  0 & (b_{12}c_-) & (b_{12}c_+ + 2J_3S) & b_{11} & b_{15}e_y & b_{16}e_y \\
  b_{15}e_x^* & b_{16}e_y & b_{16}e_y^* & b_{15}e_x & b_{55} & b_{56} \frac{c_x + c_y}{2} \\
  b_{16}e_x & b_{15}e_y^* & b_{15}e_y & b_{16}e_x^* & b_{56} \frac{c_x + c_y}{2} & b_{55}
\end{pmatrix}.
\]

\( (25b) \)
where

\[ e_x = \exp(\imath q_x a/2), \quad e_y = \exp(\imath q_y a/2), \quad c_x = \cos(q_x a), \quad c_y = \cos(q_y a) \]

\[ c_+ = \cos[a(q_x + q_y)/2], \quad c_- = \cos[a(q_x - q_y)/2], \quad c_z = \cos(q_z c/2). \tag{26} \]

From now on we will analyze the energies of the modes for wave vectors of the form \( \mathbf{G} + q_z \hat{z} \), where \( \mathbf{G} \) is a reciprocal lattice vector. In that case the matrices \( \mathbf{A} \) and \( \mathbf{B} \) can be brought into block diagonal form consisting of three \( 2 \times 2 \) blocks. The unitary transformation such that \( \mathbf{U}^\dagger \mathbf{A} \mathbf{U} \) and \( \mathbf{U}^\dagger \mathbf{B} \mathbf{U} \) are block diagonal depends on \( \mathbf{G} \), although, of course, the mode energies do not. For \( \mathbf{G} = 0 \) we have

\[ \mathbf{U} = \begin{pmatrix} 1/\sqrt{2} & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 1/\sqrt{2} & 1/2 & 0 & -1/2 & 0 \\ 0 & -1/\sqrt{2} & 1/2 & 0 & -1/2 & 0 \\ -1/\sqrt{2} & 0 & 1/2 & 0 & 1/2 & 0 \\ 0 & 0 & 0 & 1/\sqrt{2} & 0 & 1/\sqrt{2} \\ 0 & 0 & 0 & 1/\sqrt{2} & 0 & -1/\sqrt{2} \end{pmatrix}. \tag{27} \]

\( \mathbf{U}(\mathbf{G}) \) for general \( \mathbf{G} \) is given in Appendix A. For \( \mathbf{G} = 0 \) the transformed block-diagonal matrices corresponding to columns 1 and 2, (labeled "12"), those for columns 3 and 4 (labeled \( \sigma = +1 \)), and those for columns 5 and 6 (labeled \( \sigma = -1 \)) are

\[ \mathbf{A}_{12} = \begin{pmatrix} a_{11} & 0 \\ 0 & a_{11} \end{pmatrix}, \tag{28a} \]

\[ \mathbf{A}_{\sigma} = \begin{pmatrix} a_{11} + 2\sigma a_{12} & \sqrt{2}(a_{15} + \sigma a_{16}) \\ \sqrt{2}(a_{15} + \sigma a_{16}) & a_{55} + \sigma a_{56} \end{pmatrix}, \tag{28b} \]

\[ \mathbf{B}_{12} = \begin{pmatrix} b_{11} & 2J_3 S c_z \\ 2J_3 S c_z & b_{11} \end{pmatrix}, \tag{29a} \]

\[ \mathbf{B}_{\sigma} = \begin{pmatrix} b_{11} + 2\sigma b_{12} + 2\sigma J_3 S c_z & \sqrt{2}(b_{15} + \sigma b_{16}) \\ \sqrt{2}(b_{15} + \sigma b_{16}) & b_{55} + \sigma b_{56} \end{pmatrix}. \tag{29b} \]

These results remain valid when \( \mathcal{H}_2 \) is included, providing it is evaluated at zero wave vector, which, as we have said, is an excellent approximation.
D. Isotropic Interactions

For a qualitative understanding of the mode structure we start by considering the results of linearized spin-wave theory when all exchange interactions are isotropic. Then one has

\[ a_{11} = 4JS + 2J_3S, \quad a_{16} = b_{15} = J_{12}S, \quad a_{55} = b_{56} = 4J_2S, \quad b_{12} = 2JS \]  

(30)

and all the other matrix elements are zero.

In the ”12” sector, we find two optical modes which are degenerate for all \( q_z \), with

\[ (\omega/S)^2 = (4J + 2J_3)^2 - (2J_3c_z)^2 \approx 16J^2. \]  

(31)

Spin-wave interactions and anisotropic exchange interactions will have only negligible effects on these optical modes and accordingly we will generally not discuss these modes any further.\(^\text{19}\)  

In the \( \sigma = +1 \) sector we find modes with energies

\[ (\omega^+_{c}/S)^2 = 2J_3(1 - c_z) [8J + 2J_3(1 + c_z)] \approx 16J J_3(1 - c_z), \]

\[ \omega^+_{c} = 0. \]  

(32)

Finally, the \( \sigma = -1 \) sector has modes whose energies are

\[ (\omega^-_{c}/S)^2 = 2J_3(1 - c_z) [8J + 2J_3(1 + c_z)] \approx 16J J_3(1 - c_z), \]

\[ \omega^-_{c} = 0. \]  

(33)

Note that all modes are gapless at zero wave vector and that for both \( \sigma = +1 \) and \( \sigma = -1 \) we have a dispersionless zero frequency mode due to the frustration of the CuI - CuII interaction.

Several aspects of the above results are noteworthy. First of all, as we will see from our calculation of the dynamic structure factor in Sec. VII, the \( \sigma = +1 (\sigma = -1) \) sector corresponds to modes in which the spins move out of (within) the basal plane and therefore we will refer to these modes as out-of-plane (in-plane) modes. (This identification can also be deduced from the way the mode energies depend on the out-of-plane and in-plane anisotropies.) For both out-of-plane and in-plane modes note the existence of a completely gapless mode: when the CuI's rotate in phase, they produce zero coupling on the CuII's, each plane of which can be rotated with zero cost in energy. The higher-energy out-of-plane and in-plane modes are degenerate
because we have not yet included any anisotropy and these modes give rise to the usual twofold degenerate mode of the CuI subsystem. Even when more general anisotropic interactions are included, the higher-energy modes remain mostly on the CuI’s and the lower-energy modes remain mostly on the CuII’s.

E. Mode Energies for General Interactions

Here we give the mode energies in terms of the matrix elements of Eq. (25) for general interactions for wave vectors of the form $q = (0, 0, q_z)$. (The eigenvalues, but not the matrices, are invariant under addition of a reciprocal lattice vector $G$ to $q$.) To evaluate Eq. (24) within the low-frequency sectors $\sigma = \pm 1$, we record the form of the two by two blocks. Since we need both $A + B$ and $A - B$, we write

$$[A + \eta B]_{\sigma} = \begin{pmatrix} a_{11} + 2\sigma a_{12} + \eta b_{11} + 2\sigma \eta J_3 S c_z + 2\sigma \eta b_{12} & \sqrt{2}[a_{15} + \sigma a_{16} + \eta b_{15} + \sigma \eta b_{16}] \\ \sqrt{2}[a_{15} + \sigma a_{16} + \eta b_{15} + \sigma \eta b_{16}] & a_{55} + \sigma a_{56} + \eta b_{55} + \sigma \eta b_{56} \end{pmatrix}$$

(34)

In evaluating Eq. (24) it is useful to note that in the $\sigma = +1$ sector the matrix element $[A_{11} + B_{11}]_+ \sim 8JS$ is by far the largest matrix element. Similarly in the $\sigma = -1$ sector $[A_{11} - B_{11}]_- \sim 8JS$ is by far the largest matrix element. In either case, then, Eq. (24) gives the squares of the mode energies as the eigenvalues of a matrix (or its transpose) of the form

$$\begin{pmatrix} U & V \\ V & W \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} u \\ v \end{pmatrix}$$

(35)

where $\sqrt{U}u$ dominates all other matrix elements. In that case the eigenvalues are

$$(\omega^>)^2 = U u + 2V v + W v^2/u, \quad (\omega^<)^2 = (U W - V^2)(u v - v^2)/(\omega^>)^2.$$  

(36)

Explicitly, within the sectors $\sigma = \pm 1$, we have

$$U_\sigma = a_{11} + 2b_{12} + \sigma(2a_{12} + b_{11}) + 2J_3 S c_z,$$

$$V_\sigma = \sqrt{2}[a_{15} + b_{16} + \sigma(a_{16} + b_{15})],$$

$$W_\sigma = a_{55} + b_{56} + \sigma(a_{56} + b_{55}),$$

$$u_\sigma = a_{11} - 2b_{12} - 2J_3 S c_z + \sigma(2a_{12} - b_{11}),$$

$$v_\sigma = a_{11} - 2b_{12} - 2J_3 S c_z - \sigma(2a_{12} - b_{11}).$$
\[ v_\sigma = \sqrt{2} [a_{15} - b_{16} + \sigma (a_{16} - b_{15})] , \]
\[ w_\sigma = a_{55} - b_{56} + \sigma (a_{56} - b_{55}) . \]  

Substituting these evaluations into Eq. (36) [or, if need be, exactly implementing Eq. (24)] gives the four low energy modes for wave vectors along the \( \overline{\hline} \) direction. Obviously, since the mode energies are derived from a two by two dynamical matrix, we can easily obtain exact expressions for their energies.

IV. NONLINEAR SPIN WAVES

A. \( 1/S \) Corrections to \( J, J_3, \) and \( J_2 \)

When we include the effect of spin-wave interactions at order \( 1/S \) on the CuI-CuI interactions or on the CuII-CuII interactions, we expect to get a simple renormalization. For the exchange interactions between neighbors in the same CuO plane, this effect is well known. As explained above, we decouple the fourth order terms in \( \mathbf{S}_{ai} \cdot \mathbf{S}_{bj} \) as

\[ -\frac{1}{2} [b_j b_j a_i + b_j a_i a_i] + 2 a_i a_i b_j b_j \rightarrow -<a_i a_i + a_i b_j][a_i a_i + b_j b_j] + a_i b_j + a_i b_j \]  

and those in \( \mathbf{S}_{ei} \cdot \mathbf{S}_{fj} \) as

\[ -\frac{1}{2} [e_i f_j e_i f_j] + e_i f_j e_i f_j \rightarrow -<e_i e_i + e_i f_j][e_i e_i + f_j f_j] + e_i f_j + e_i f_j \]  

From this result we conclude that \( J \) and \( J_2 \) should be replaced by \( Z_c J \) and \( Z_2 J_2 \), respectively, with \( Z_c = 1 - (1/S)<a_i a_i + a_i b_j> \), where \( i \) and \( j \) are nearest neighboring sites on the \( a \) and \( b \) sublattices, respectively, and \( Z_2 = 1 - (1/S)<e_i e_i + e_i f_j \) in a similar notation, so that \( Z_2 \approx Z_c \) at zero temperature. \( Z_c \) has been calculated more accurately than this. (In Ref. 20 the value \( Z_c \approx 1.17 \) is given.) For \( J_3 \) we note that \( a_i \) and \( b_j \) refer to sites in different CuO planes, in which case \( <a_i b_j> \approx 0 \). So we should replace \( J_3 \) by \( \tilde{Z}_3 J_3 \), where

\[ \tilde{Z}_3 = 1 - (1/S)<a_i a_i> \]  

so that \( \tilde{Z}_3/2 \) is essentially the magnitude of the zero-point staggered spin in the presence of quantum fluctuations. (Thus \( \tilde{Z}_3 \approx 0.6 \) is very different from \( Z_c \).)
B. The Effect of Spin-Wave Interactions on $J_{12}$

Now we discuss the effect of spin-wave interactions on $J_{12}$, i.e. we consider the Shender interaction. Correctly to order $1/S$ we construct the effective quadratic Hamiltonian by contracting two operators in all possible ways. I.e. we replace two operators by the thermal expectation value (indicated by $\langle \ldots \rangle$) of their product. Applying this procedure to the relevant terms in Eq. (14) we obtain the effective interactions between a CuI spin $i$ on sublattice $g$ and nearest neighboring CuII spins as

$$V_{ae}/J_{12} = a_i^\dagger a_i \left( S - \langle a_i^\dagger e_j^\dagger \rangle - \langle e_i^\dagger e_j \rangle \right) + e_j^\dagger e_j \left( S - \langle e_j a_i \rangle - \langle a_i^\dagger a_i \rangle \right) + a_i e_j \left( S - \langle e_j^\dagger e_j \rangle - \langle a_i^\dagger e_j \rangle \right) + a_j^\dagger e_j \left( S - \langle a_i^\dagger a_i \rangle - \langle a_i e_j \rangle \right)$$

$$V_{af}/J_{12} = a_i^\dagger a_i \left( -S - \langle a_i^\dagger f_j^\dagger \rangle + \langle f_j^\dagger f_j \rangle \right) + f_j^\dagger f_j \left( -S - \langle a_i f_j^\dagger \rangle + \langle a_i^\dagger a_i \rangle \right) + a_i f_j \left( S - \langle a_i^\dagger a_i \rangle + \langle a_i f_j \rangle \right) + f_j^\dagger a_i \left( S - \langle f_j^\dagger f_j \rangle + \langle a_i^\dagger f_j \rangle \right).$$

(41)

(42)

Here to leading order in $1/S$ it suffices to evaluate the various expectation values with respect to the original quadratic Hamiltonian. At quadratic order we have symmetry such that $\langle a_i^\dagger e_j^\dagger \rangle = \langle a_i e_j \rangle$, $\langle e_i^\dagger e_j \rangle = \langle f_j^\dagger f_j \rangle$, etc. We define

$$J_{12}^{(1)} S/J_{12} = S + \langle a_i f_j^\dagger \rangle - \langle a_i^\dagger a_i \rangle$$

$$J_{12}^{(2)} S/J_{12} = S + \langle a_i f_j^\dagger \rangle - \langle f_j^\dagger f_j \rangle$$

$$J_{12}^{(3)} S/J_{12} = S - \langle a_i e_j \rangle - \langle e_i^\dagger a_i \rangle$$

$$J_{12}^{(4)} S/J_{12} = S - \langle a_i e_j \rangle - \langle e_i^\dagger e_j \rangle.$$ 

(43)

Note that $J_{12}^{(3)} - J_{12}^{(4)} = J_{12}^{(1)} - J_{12}^{(2)}$. Then

$$V_{ae} = J_{12}^{(4)} S a_i^\dagger a_i + J_{12}^{(3)} S e_j^\dagger e_j + J_{12}^{(4)} S a_i e_j + J_{12}^{(3)} S a_i^\dagger e_j$$

$$V_{af} = -J_{12}^{(2)} S a_i^\dagger a_i - J_{12}^{(1)} S f_j^\dagger f_j + J_{12}^{(1)} S a_i^\dagger f_j + J_{12}^{(2)} S f_j^\dagger a_i.$$ 

(44)

Since we only work to order $1/S$, we keep only the Hermitian part of these perturbations:

$$V_{ae} = J_{12}^{(4)} S a_i^\dagger a_i + J_{12}^{(3)} S e_j^\dagger e_j + J_{12}^{(3)} S \left( a_i e_j + a_i^\dagger e_j^\dagger \right)$$

$$V_{af} = -J_{12}^{(2)} S a_i^\dagger a_i - J_{12}^{(1)} S f_j^\dagger f_j + J_{12}^{(2)} S \left( a_i^\dagger f_j + f_j^\dagger a_i \right).$$ 

(45)
where
\[
J_{12}^{(12)} = \frac{1}{2}[J_{12}^{(1)} + J_{12}^{(2)}], \quad J_{12}^{(34)} = \frac{1}{2}[J_{12}^{(3)} + J_{12}^{(4)}].
\] (46)

As it turns out, the energies of the modes we study depend only on the single parameter
\[
\alpha = (J_{12}^{(4)} - J_{12}^{(2)})S = (J_{12}^{(3)} - J_{12}^{(1)})S = -J_{12}(\langle a_i e_j \rangle + \langle a_i f_j^{\dagger} \rangle).
\] (47)

Note that the parameter \( \delta \) in Ref. 5 is \( \delta = \alpha / S \). We evaluate this parameter in Appendix B and find
\[
\alpha = C_\alpha J_{12}^2 / J,
\] (48)

where \( C_\alpha \) is a numerical factor which we found to be 0.1686. The anharmonic effects of Eq. (45) give rise to contributions to the dynamical matrix of
\[
\begin{align*}
\delta a_{11} &= \alpha, \\
\delta a_{16} &= J_{12}^{(12)} S - J_{12} S, \\
\delta a_{55} &= 2\alpha, \\
\delta b_{15} &= J_{12}^{(34)} S - J_{12} S.
\end{align*}
\] (49)

It is known\textsuperscript{8,21,22} that in simpler problems these anharmonic effects give rise at zero momentum to effective biquadratic exchange interactions between sublattices which otherwise are frustrated in harmonic theory. To emphasize this point we treat a biquadratic interaction between nearest CuI - CuII neighbors (in the plane) which is of the form
\[
H_{BQ} = -\frac{j_{BQ}}{S^2} \sum_{i \in I} \sum_{\vec{\delta}_{2,1}} (\mathbf{S}_i \cdot \mathbf{S}_{i+\vec{\delta}_{2,1}})^2.
\] (50)

Then the contributions to the dynamical matrix are
\[
\begin{align*}
\delta a_{11} &= 4j_{BQ} S, \\
\delta a_{16} &= -2j_{BQ} S, \\
\delta a_{55} &= 8j_{BQ} S, \\
\delta b_{15} &= 2j_{BQ} S.
\end{align*}
\] (51)
Then using Eqs. (37) and (36) we find the mode energies at zero transverse wave vector (for large $J$) are now

$$
(\omega^>_{\sigma})^2 = 8JS[\alpha + 4j_{BQ}S + 2J_3S(1 - c_z)] = 8JS[\alpha_{\text{eff}} + 2J_3S(1 - c_z)],
$$

$$
(\omega^<_{\sigma})^2 = \frac{4\alpha_{\text{eff}}J_3S(1 - c_z)(8J_2S + \alpha_{\text{eff}})}{\alpha_{\text{eff}} + 2J_3S(1 - c_z)},
$$

(52)

where

$$
\alpha_{\text{eff}} = \alpha + 4j_{BQ}S.
$$

These results demonstrate that the Shender interaction does mimic a biquadratic exchange interaction at long wave length. However, in view of the relation for spin 1/2 that $(S_i \cdot S_j)^2 = \frac{3}{16} - \frac{1}{2}S_i \cdot S_j$, a biquadratic exchange interaction between two spins 1/2 is equivalent to a Heisenberg exchange interaction, and we may therefore assume that $j_{BQ}$ vanishes.

As before, there is degeneracy between in-plane and out-of-plane energies because we have not yet included anisotropy. However, by taking into account spin-wave interactions we now have the mode structure one would expect for an isotropic antiferromagnet: We have a doubly degenerate zero energy Goldstone mode at zero wave vector, and doubly degenerate nonzero energy modes for zero wave vector as shown in the right-hand panel of Fig. 3. The quantum gap in the optical mode $\omega^>_{\sigma}$ at zero wave vector has been obtained for a number of other frustrated systems in several theoretical studies\textsuperscript{23,21,22} beginning with the work of Shender.\textsuperscript{8} However, because we have two subsystems which order at different temperatures, the emergence of this gap has a very unique signature not present in other experimental systems studied up to now.\textsuperscript{9}

V. INCLUSION OF ANISOTROPIES

A. Out – of – Plane Exchange Anisotropy

To obtain the correct energy gaps at zero wave vector we must add the anisotropy due to anisotropic exchange interactions. (Since we are dealing with spin 1/2’s, there can be no single ion anisotropy.) In this subsection we include out–of–plane exchange anisotropy. This part of the anisotropic exchange energy between sublattices $a$ and $b$ of the CuI’s is given as
\[ V_{ab} \equiv -\Delta J_1 \sum_{i \in a,j \in b} S_{ai}^z S_{bj}^z \Delta_{ij} , \]  

where \( \Delta_{ij} \) is defined so as to implement the nearest neighbor restriction. Thus, neglecting anharmonicity, we write

\[
V_{ab} = \frac{1}{4} \Delta J_1 \sum_{i \in a,j \in b} (S_{ai}^+ - S_{ai}^-)(S_{bj}^+ - S_{bj}^-) \Delta_{ij} \nonumber
\]

\[
= \frac{1}{2} \Delta J_1 S \sum_{i \in a,j \in b} (a_i - a_i^\dagger)(b_j^\dagger - b_j) \Delta_{ij} \nonumber
\]

\[
= \Delta J_1 S \sum_q [a^\dagger(q)b(q) + b(q)a(q) - a^\dagger(q)b^\dagger(-q) - a(q)b(-q)]c_+ . \]  

This result allows us to identify the contribution to the parameters of the dynamical matrix introduced in Eq. (25) as

\[
\delta a_{12} = \Delta J_1 S , \quad \delta b_{12} = -\Delta J_1 S , 
\]

without having to explicitly consider the other CuI-CuI interactions.

Next we consider the out-of-plane anisotropy of the CuI - CuII interactions. From the form of Eq. (25) we see that we only need construct the a-e and a-f interactions. For the a-e interaction we have

\[
V_{ae} = -\Delta J_{12} \sum_{i \in a,j \in e} S_{ai}^z S_{ej}^z \Delta_{ij} = \frac{1}{4} \Delta J_{12} \sum_{i \in a,j \in e} \left(S_{ai}^+ - S_{ai}^-\right) \left(S_{ej}^+ - S_{ej}^-\right) \Delta_{ij} \nonumber
\]

\[
= \frac{1}{2} \Delta J_{12} S \sum_{i \in a,j \in e} \left(a_i - a_i^\dagger\right) \left(e_j^\dagger - e_j\right) \Delta_{ij} \nonumber
\]

\[
= \frac{1}{2} \Delta J_{12} S \sum_q \left[a(q)e^\dagger(q)e^{i\alpha(r_e - r_a)} - a^\dagger(-q)e^\dagger(q)e^{i\alpha(r_a - r_e)} + h.c.\right] \nonumber
\]

\[
= \frac{1}{2} \Delta J_{12} S \sum_q \left[a(q)e^\dagger(q)e_x^* - a^\dagger(-q)e^\dagger(q)e_x^* + a^\dagger(q)e(q)e_x - a(-q)e(q)e_x\right] , \]  

which gives a contribution to the dynamical matrix with

\[
\delta a_{15} = \frac{1}{2} \Delta J_{12} S , \quad \delta b_{15} = -\frac{1}{2} \Delta J_{12} S . \]  

Similarly

\[
V_{af} = \frac{1}{2} \Delta J_{12} S \sum_q \left[a(q)f(-q)e_x - a^\dagger(q)f(q)e_x^* + a^\dagger(q)f^\dagger(-q)e_x^* - a(q)f^\dagger(q)e_x\right] , \]  

21
from which we deduce that

\[ \delta a_{16} = -\frac{1}{2} \Delta J_{12} S, \quad \delta b_{16} = \frac{1}{2} \Delta J_{12} S. \]  

(60)

Finally we include the out-of-plane anisotropy of the CuII-CuII interactions. Thus

\[ V_{ef} = -\Delta J_2 \sum_{i \in e, j \in f} S_{ei}^z S_{fj}^z \Delta_{ij} \]

\[ = \frac{1}{4} \Delta J_2 \sum_{i \in e, j \in f} |S_{ei}^+ - S_{ei}^-| |S_{fj}^+ - S_{fj}^-| \Delta_{ij} \]

\[ = \frac{1}{2} \Delta J_2 S \sum_{i \in e, j \in f} (e_i - e_i)(f_j - f_j') \Delta_{ij} \]

\[ = \Delta J_2 S \sum_{q} \left( e^{\dagger}(q)f(q) - e^{\dagger}(q)f^{\dagger}(-q) - e(q)f(-q) + f^{\dagger}(q)e(q) \right) [e_x + c_y], \]

(61)

which leads to

\[ \delta a_{36} = 2\Delta J_2 S, \quad \delta b_{36} = -2\Delta J_2 S. \]

(62)

The renormalization (at order 1/S) of the out-of-plane anisotropy is accomplished by replacing \( \sqrt{J \Delta J_1} \) by \( Z_g \sqrt{J \Delta J_1}. \)

It is instructive to see the influence of this anisotropy on the gaps at zero wave vector. Referring to Eq. (36) we see that the high energy mode gap due to the Shender fluctuation term, causes \( U u \) to be nonzero. To check for gaps in the mode energies \( \omega_{\sigma} \) at zero wave vector it suffices to consider the quantity

\[ \Lambda \equiv uw - v^2 = [2\Delta J_1 S(1 + \sigma) + \alpha][2\alpha + 2\Delta J_2 S(1 + \sigma)] - [-\sqrt{2\sigma\alpha}]^2. \]

(63)

When we turn off both out-of-plane anisotropies, \( \Delta J_1 \) and \( \Delta J_2 \), the two modes \( \omega_{\sigma} \) are gapless. When we allow the out-of-plane anisotropy to be nonzero, we clearly introduce a gap (\( \Lambda \) is nonzero) in the out-of-plane (\( \sigma = 1 \)) sector but not in the in-plane (\( \sigma = -1 \)) sector. This result follows from the fact that the spins can still undergo a global rotation within the easy plane at no cost in energy. Hence we still have a single Goldstone mode with zero energy at zero wave vector. In order for this mode to have a gap, we have to take account of effects which lead to a fourfold in-plane anisotropy which we consider in the next subsection.
B. In – Plane Exchange Anisotropy

1. CuI-CuI Interactions

In this subsection we discuss the effects of the in–plane anisotropy of the CuI-CuI exchange interactions. First of all, note that this perturbation is extremely weak. It gives rise to an effective fourfold anisotropy. This very small fourfold anisotropy only has a nonnegligible effect within the low frequency sector and even there only at zero wave vector. The Hamiltonian describing the in–plane anisotropy of the CuI-CuI interactions is

\[ V_{\text{in}} = \delta J_1 \sum_{i \in a, d; \delta} \sigma(\delta) \left( S_i^x S_{i+\delta}^x - S_i^y S_{i+\delta}^y \right), \]  

(64)

where \( j = i + \delta, \delta \) is summed over four values (the two \( \delta_+ \)'s and the two \( \delta_- \)'s), and \( \sigma(\delta_{\pm}) = \pm 1 \). Then

\[ V_{\text{in}} = \delta J_1 \sum_{i \in a, d; \delta} \sigma(\delta) \left[ \left( S - \alpha_i^1 \alpha_i(-S + \beta_j^1 \beta_j) - \frac{1}{4}(2S) \left( \alpha_i^1 \phi(\alpha_i) \right) \left( \beta_j^1 + \phi(\beta_j) \beta_j \right) \right] \]

\[ = \delta J_1 \sum_{i \in a, d; \delta} \sigma(\delta) \left[ -\alpha_i^1 \beta_j^1 \beta_j - \frac{1}{2} S[\alpha_i^1 + \alpha_i^1 [\beta_j^1 + \beta_j] + \frac{1}{4} \alpha_i^1 \alpha_i^1 \alpha_i^1 \beta_j^1 \beta_j + \frac{1}{8} \delta \alpha_i^1 \alpha_i^1 \beta_j^1 \beta_j \right], \]  

(65)

where \( \alpha_i = a \) if site \( i \) is an \( a \) site and \( \alpha_i = d \) if \( i \) is a \( d \) site, and similarly for \( \beta_j \). We write

\[ V_{\text{in}} = V_{2,\text{in}} + V_{4,\text{in}} + V_{6,\text{in}}, \]  

(66)

where the subscript 2 (4 or 6) indicates terms quadratic (fourth or sixth) order in boson operators. Since we work systematically to first order in \( 1/S \), we neglect \( V_{6,\text{in}} \). Also

\[ V_{2,\text{in}} = -\frac{1}{2} \delta J_1 S \sum_{\delta, k} \sigma(\delta) (\alpha_i + \alpha_i^2) (\beta_j + \beta_j^2) \]

\[ = -\delta J_1 S \sum_{\delta, k} \left( (a^\dagger(k) + a(-k)) [b(k) + b^\dagger(-k)] + [d^\dagger(k) + d(-k)] [c(k) + c^\dagger(-k)] \right) \]

\[ + \delta J_1 S \sum_{\delta, k} \left( (a^\dagger(k) + a(-k)) [c(k) + c^\dagger(-k)] + [d^\dagger(k) + d(-k)] [b(k) + b^\dagger(-k)] \right) \]  

(67)

and
\[ V_{4,\text{in}} = \delta J_1 \sum_{i \in a, d} \sigma(\delta) \left( -\alpha_i \beta_j \beta_j + \frac{i}{4} \alpha_i^4 \alpha_i (\beta_j^4 + \beta_j) + \frac{1}{4} (\alpha_i^4 + \alpha_i) \beta_j^4 \beta_j \right). \]  

(68)

We now consider the effect of \( V_{2,\text{in}} \) on the spectrum for \( k_x = k_y = 0 \), so that \( c_+ = c_- = 1 \). In this case because the perturbation is proportional to \( b - c \) or to \( b^\dagger - c^\dagger \), one sees that \( V_{2,\text{in}} \) only couples to the optical mode sector. Accordingly, we do not consider \( V_{2,\text{in}} \) any further.

We expect that this in-plane anisotropy should give rise to a macroscopic four-fold anisotropy. In order to obtain this anisotropy we must include anharmonic effects at relative order \( 1/S \). Now we decouple the four operator terms into quadratic terms times averages of the remaining quadratic factors. This calculation is done in Appendix C. In that calculation we naturally drop all contributions to the optical mode sector and of the rest keep only terms which have an effect on the mode energies at zero wave vector. The result is that contributions to the dynamical matrices due to the in-plane CuI – CuI interactions yield

\[
\begin{align*}
\delta a_{11} &= 16C_2 \tau \\
\delta a_{12} &= -4(6C_2 - C_{2c} - 4C_{2b}) \tau \\
\delta b_{11} &= 8C_{2c} \tau \\
\delta b_{12} &= -16C_{2b} \tau ,
\end{align*}
\]

(69a)  
(69b)  
(69c)  
(69d)

where \( \tau \equiv (\delta J_1)^2/J \) and the \( C \)'s are lattice sums defined in Eq. (C19) of Appendix C. It turns out that because \( \tau \) is so small, the only evaluation we need is that \( C_2 = 0.01 \). Note that the contributions in Eq. (69) are of relative order \( 1/S \) which is consistent with the fact that they represent the effect of quantum fluctuations. The fact that they represent a modification in the zero-point energy is reflected by the appearance of the factor \( C_2 \ll 1 \).

2. CuI-CuII Interactions

Next we deal with the in-plane anisotropy of the CuI-CuII interactions. The terms in Eq. (12) involving \( \delta J_{12} \) are

\[
\begin{align*}
V_{\delta J_{12}} &= -\delta J_{12} \sum_{i \in II, \delta_z} \left[ S_i^{\xi^*} S_{i+\delta_z}^\eta + S_i^\eta S_{i+\delta_z}^{\xi^*} \right] + \delta J_{12} \sum_{i \in II, \delta_y} \left[ S_i^{\xi^*} S_{i+\delta_y}^\eta + S_i^\eta S_{i+\delta_y}^{\xi^*} \right].
\end{align*}
\]

(70)

In terms of boson operators this is

\[ \text{24} \]
\[
V_{\delta J_{12}} = \delta J_{12} \sqrt{S/2} \sum_i \left[ \left( S - e_i e_i \right) \left( a_i + a_i^\dagger + d_i + d_i^\dagger \phi(d_{i-x}) \right) + \left( e_i^\dagger + \phi(e_i)e_i \right) \left( -2S + a_i^\dagger a_i + \phi(d_{i-x}) \right) \right]
\]
\[
- \delta J_{12} \sqrt{S/2} \sum_i \left[ \left( S - e_i e_i \right) \left( b_i^\dagger + \phi(b_{i-y})b_{i-y} + c_i^\dagger + \phi(c_{i+y})c_{i+y} \right) + \left( e_i^\dagger + \phi(e_i)e_i \right) \left( 2S - b_i^\dagger b_{i-y} - c_i^\dagger + \phi(c_{i-y})c_{i-y} \right) \right]
\]
\[
- \delta J_{12} \sqrt{S/2} \sum_i \left[ \left( S + f_i^\dagger f_i \right) \left( a_i + a_i^\dagger + d_i + d_i^\dagger \phi(d_{i-x}) \right) + \left( f_i + f_i^\dagger \phi(f_i) \right) \left( 2S - a_i^\dagger a_i + \phi(d_{i-x}) \right) \right]
\]
\[
+ \left( f_i + f_i^\dagger \phi(f_i) \right) \left( -2S + a_i^\dagger a_i + \phi(d_{i-x}) \right)
\]
\[
\left( S + f_i^\dagger f_i \right) \left( b_i^\dagger + \phi(b_{i-y})b_{i-y} + c_i^\dagger + \phi(c_{i-y})c_{i-y} \right) + \left( f_i + f_i^\dagger \phi(f_i) \right) \left( 2S - b_i^\dagger b_{i+y} - c_i^\dagger + \phi(c_{i-y})c_{i-y} \right)
\].

(71)

This perturbation contains terms linear and terms cubic in the boson operators. The linear terms and (at relative order 1/S) the cubic terms will shift the equilibrium so that the boson operators are modified as
\[
e_i \rightarrow e_i + s, \quad f_i \rightarrow f_i + s, \quad a_i \rightarrow a_i + t, \quad b_i \rightarrow b_i + t, \quad c_i \rightarrow c_i + t, \quad d_i \rightarrow d_i + t.
\]

(72)

These shifts are evaluated in Appendix D, where we find that (to leading order in 1/S)
\[
s = \frac{4\delta J_{12} \sqrt{S/2}}{8J_2}, \quad t = -\frac{2J_{12} s}{8J + 4J_3} = -\frac{J_{12} \delta J_{12} \sqrt{S/2}}{J_2(8J + 4J_3)}.
\]

(73)

These are the expected results. As one sees from Eq. (12), the perpendicular field acting on an e spin is \(4\delta J_{12} s\) in the positive \(\eta\) direction, so that the perpendicular moment of the e spin is \(\Delta S_e = 4\delta J_{12} S \chi_{II} = 4\delta J_{12} S/(8J_2)\), which agrees with \(\sqrt{2S}s\) when Eq. (73) is used. Further, due to the isotropic exchange, the field acting on an a spin is \(2J_{12} \Delta S_a = J_{12} \delta J_{12} S/J_2\) in the negative \(\eta\) direction. Thus \(\Delta S_a = -[J_{12} \delta J_{12} s/J_2] \chi_I = -[J_{12} \delta J_{12} s/J_2]/[8J + 4J_3],\) which agrees with \(\sqrt{2S}t\) when Eq. (73) is used. Note that \(\Delta S_e\) and \(\Delta S_a\) are both of order \(S\), a result which indicates that the effects here are completely classical.

To determine the effect of \(V_{\delta J_{12}}\) on the spin-wave spectrum we need to construct the effective quadratic Hamiltonian, which results from introducing shifts into anharmonic terms. This is

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done in Appendix D. When we insert these shifts into the cubic terms of $V_{\delta J_{12}}$ we ignore $t$ in comparison to $s$ because $J \gg J_{12}$. Thereby we get contributions to the dynamical matrix of

$$
\delta a_{55} = \delta a_{11} = 2\delta b_{55} = \delta J_{12}^2 S / J_2 \equiv \zeta S ,
$$
$$
\delta a_{16} = -\delta a_{15} = \delta b_{16} = -\delta b_{15} = \frac{1}{4}\zeta S .
$$

(74)

We also insert these shifts into the four operator terms of the isotropic Hamiltonian. As before we only keep terms arising from replacing two CuII operators by $\langle e \rangle$. The magnitude of other terms, e.g. CuI - CuI quartic terms when CuI shifts $\langle a \rangle$ are kept, are shown in Appendix D to be much smaller than those we have kept. The result of the calculation in Appendix D is that we get the contributions to the dynamical matrix of

$$
\delta a_{55} = -\zeta S , \quad \delta b_{55} = -\frac{1}{4}\zeta S ,
$$
$$
\delta a_{56} = -\frac{3}{4}\zeta S , \quad \delta b_{56} = -\zeta S .
$$

(75)

Note that these perturbative contributions from the CuI-CuII in-plane anisotropy, are proportional to $S$, unlike the case for the other in-plane anisotropies. This indicates that the effect of $\delta J_{12}$ (which we called $J_{pd}$ previously$^{3,4}$), is a classical effect which already appeared within mean field theory.$^{3,4}$ The other in-plane anisotropies only have an effect when we consider fluctuations. However, since the effect of $\delta J_{12}$ is rather small, we do not consider the effects of fluctuation corrections to it.

### 3. CuII-CuII Intraplanar Interactions

Here we consider the in-plane anisotropy of the interactions between pairs of CuII spins in the same plane. Their interaction is

$$
V = -\delta J_2 \sum_{i \in e} \left\{ \sum_{\delta_{j} = i + 2\delta_x} S_i^z S_j^z + S_i^y S_j^y \right\} + \sum_{\delta_{j} = i + 2\delta_y} \left\{ \delta J_2 \sqrt{\frac{S}{2}} \sum_{i \in \delta_y} \left\{ (S - e_i^\dagger e_i) \left[ f_j + f_j^\dagger \phi(f_j) \right] + \left[ e_i^\dagger + \phi(e_i)e_i \right] (-S + f_j^\dagger f_j) \right\} + \right. \\
+ \delta J_2 \sqrt{\frac{S}{2}} \sum_{i \in \delta_y} \left\{ (S - e_i^\dagger e_i) \left[ f_j + f_j^\dagger \phi(f_j) \right] + \left[ e_i^\dagger + \phi(e_i)e_i \right] (-S + f_j^\dagger f_j) \right\}
$$

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\[
\delta J_2 \sqrt{\frac{S}{2}} \left[ \sum_{i,j} \left( e_i^+ e_i (f_j + f_j^+) - (e_i^+ + e_i) f_j^+ f_j \right) - \sum_{i,j} \left( e_i^+ e_i (f_j + f_j^+) - (e_i^+ + e_i) f_j^+ f_j \right) \right]
\]

\[
= \delta J_2 \sqrt{\frac{8S}{N}} \sum_{q,k} \rho(k) \left( [f(k) + f^\dagger(-k)]e^\dagger(q)e(q - k) - [e(k) + e^\dagger(-k)]f^\dagger(q)f(q - k) \right),
\]

where

\[
\rho(k) = \frac{1}{2} \left[ \cos(ak_x) - \cos(ak_y) \right].
\]

This Hamiltonian is treated in Appendix E, where the additional contributions to the spin-wave matrices (at \( q_z = 0 \)) are found to be

\[
\delta a_{55} = -16[\delta J_2^2 / J_2][2C_{2a} + C_{2b}] \equiv -16\xi[2C_{2a} + C_{2b}] , \quad \delta a_{56} = 16\xi[2C_{2a} - C_{2b}]
\]

and

\[
\delta b_{55} = -16\xi C_{2b} , \quad \delta b_{56} = 48\xi C_{2b},
\]

where \( C_{2a} \) and \( C_{2b} \) are lattice sums defined in Appendix C.

It is interesting to note that apart from a minus sign, these results are exactly the same as in Yildirim et al.\(^\text{13}\). This difference in sign is to be expected because the \( \text{CuII}'s \) are oriented in a hard direction with respect to only \( \text{CuII-CuII} \) interactions. Consequently, this term tends to decrease the gap.

### 4. \text{CuII-CuII} Interplanar Interactions

Here we consider the effect of interactions between a pair of \( \text{CuII} \) spins in adjacent planes. The situation we consider is shown in the left panel of Fig. 4, where one sees that the isotropic component of the \( \text{CuII-CuII} \) interplanar interaction is frustrated. To describe the anisotropy of this interaction we introduce the principal axes (shown in the right panel of Fig. 4) as follows

\[
\hat{n}_1 = (-\hat{x} + \hat{y})/\sqrt{2} , \quad \hat{n}_2 = (\hat{x} + \hat{y}) \cos \psi / \sqrt{2} + \hat{z} \sin \psi , \quad \hat{n}_3 = (\hat{x} + \hat{y}) \sin \psi / \sqrt{2} - \hat{z} \cos \psi .
\]

The angle \( \psi \) is not fixed by symmetry. We then write the anisotropic \( \text{CuII-CuII} \) interaction \( H_{ij}^{II-II} \) between nearest-neighboring spins \( i \) and \( j \) in adjacent planes as\(^{10}\)
\[ H_{ij}^{III-II} = \sum_{k=1}^{3} K_k |S_i \cdot \hat{n}_k^{(ij)}||S_j \cdot \hat{n}_k^{(ij)}|, \]  

(81)

where \( \hat{n}_k^{(ij)} \) is the \( k \)th principal axes for the pair \( ij \) which can be obtained from the right panel of Fig. 4, by a rotation of coordinates, if necessary, and \( K_k \) is the associated principal value of the exchange tensor. The contributions of this interaction to the dynamical matrix are evaluated for \( q_x = q_y = 0 \) in Appendix F as

\[ \begin{align*}
\delta a_{55} &= \delta a_{66} = 4(K_1 - K_2 c^2 - K_3 s^2)S + 2(K_1 + K_3 c^2 + K_2 s^2)S c_z, \\
\delta a_{56} &= \delta a_{65} = 2(K_2 - K_3)(c^2 - s^2)S c_z, \\
\delta b_{55} &= \delta b_{66} = 2(K_1 - K_2 s^2 - K_3 c^2)S c_z, \\
\delta b_{56} &= \delta b_{65} = 2(K_2 + K_3)c_z,
\end{align*} \]

(82a-d)

where \( c \equiv \cos \psi \) and \( s \equiv \sin \psi \). As we will see later, this interaction can only contribute significantly to the lowest energy in-plane mode, where its effect is through the combination

\[ \delta(a_{55} - a_{56} + b_{55} - b_{56}) = 4S(K_1 - K_2 c^2 - K_3 s^2)(1 + c_z) \equiv 4\Delta KS(1 + c_z). \]

(83)

Note that \( \Delta K = 0 \) for isotropic exchange.

A closely related interaction is the long-range dipolar interaction, whose contributions to the dynamical matrix are also evaluated in Appendix F. This interaction is dominant in Sr\(_2\)CuO\(_2\)Cl\(_2\)\(^{16}\). To include dipolar interactions we obtain (in Appendix F) the result

\[ \delta(a_{55} - a_{56} + b_{55} - b_{56}) = 6g^2 \mu_B^2 S(1 + c_z)X, \]

(84)

where \( X \) is the lattice sum

\[ X = \sum_{j \in II: z_i = z_j} \frac{x_{ij} y_{ij} \sigma_j}{r_{ij}^5}, \]

(85)

where \( i \) labels a fixed CuII site, \( \sigma_j \) is +1 if spins \( i \) and \( j \) are parallel and is −1 if they are antiparallel. Numerical evaluation yields

\[ X = 7 \times 10^{-4} \text{Å}^{-3}. \]

(86)

Therefore we should replace \( \Delta K \) by

\[ \Delta K_{\text{eff}} = \Delta K + \frac{3}{2}g^2 \mu_B^2 X. \]

(87)

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5. Cu-I-Cu II Interplanar Interactions

Here we briefly summarize the results for a similar treatment of the Cu-I-Cu II anisotropic interactions. The situation we consider is shown in the left panel of Fig. 5, where one sees that the isotropic component of the Cu-I-Cu II interplanar interaction is frustrated. To describe the anisotropy of this interaction we introduce the principal axes for the Cu-I-Cu II pair \( \mathbf{a} - \mathbf{c} \) shown in the right panel of Fig. 5, as follows

\[
\hat{m}_1 = -\hat{y} , \quad \hat{m}_2 = \hat{z} \cos \phi - \hat{x} \sin \phi , \quad \hat{m}_3 = -\hat{z} \sin \phi - \hat{x} \cos \phi .
\]  

(88)

The angle \( \phi \) is not fixed by symmetry. We then write the anisotropic Cu-I-Cu II interaction \( \mathcal{H}_{ij}^{I-II} \) between nearest-neighboring spins \( i \) and \( j \) in adjacent planes as

\[
\mathcal{H}_{ij}^{I-II} = \sum_{k=1}^{3} K'_k | S_i \cdot \hat{m}_k(ij) | | S_j \cdot \hat{m}_k(ij) |
\]

(89)

where \( \hat{m}_k(ij) \) is the \( k \)th principal axes for the pair \( ij \) which can by obtained from the right panel of Fig. 5, by a rotation of coordinates, if necessary, and \( K'_k \) is associated principal value of the exchange tensor. In Appendix G we obtained the following contributions to the dynamical matrices for \( q_x = q_y = 0 \)

\[
\delta a_{15} = \delta b_{16} = \frac{1}{2} \left[ K'_1 + K'_2 (1 - 3c^2) + K'_3 (1 - 3s^2) \right] \equiv G_{I-II}
\]

(90a)

\[
\delta a_{16} = \delta b_{15} = \frac{1}{2} \left[ K'_1 + K'_2 (1 + c^2) + K'_3 (1 + s^2) \right] \equiv H_{I-II}
\]

(90b)

where \( c \equiv \cos \phi \) and \( s \equiv \sin \phi \). We will see later that these terms have a negligible effect on the spin-wave spectrum.

VI. SPIN–WAVE SPECTRUM

Explicitly, the dynamical matrices corresponding to the effective quadratic Hamiltonian containing the above-mentioned anisotropies are of the form of Eq. (25) with

\[
a_{11} = 4JS + 2J_3 S + 16C_2 \tau + \zeta S + \alpha ,
\]

\[
a_{12} = \Delta J_1 S - 4(6C_2 - C_2 - 4C_{2b}) \tau ,
\]

\[
a_{44} = 8(6C_2 - C_2 - 4C_{2b}) \tau + \frac{4}{3} \Delta J_1 S - \Delta J_1 S ,
\]

\[
a_{45} = \Delta J_1 S - \Delta J_1 S ,
\]

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a_{45} = \Delta J_1 S - \Delta J_1 S ,
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\[
a_{45} = \Delta J_1 S - \Delta J_1 S ,
\]

\[
29
\[ a_{15} = \frac{1}{2} \Delta J_{12} S - \frac{1}{4} \zeta S + G_{I-II}, \]
\[ a_{16} = J_{12}^{(12)} S - \frac{1}{2} \Delta J_{12} S + \frac{1}{4} \zeta S + H_{I-II}, \]
\[ a_{55} = 4J_2 S - 16\xi(2C_2 - C_{2b}) + 2\alpha 
+ 4(K_1 - K_2 c^2 - K_3 s^2)S + 2(K_1 + K_3 c^2 + K_2 s^2)S c_z, \]  
\[ a_{56} = 2\Delta J_2 S + 16\xi(2C_2 - 3C_{2b}) - \frac{3}{4} \zeta S + 2(K_2 - K_3)(c^2 - s^2)S c_z, \]
\[ b_{11} = 8C_{2c} \tau, \]
\[ b_{12} = 2JS - \Delta J_2 S - 16C_{2c} \tau, \]
\[ b_{15} = J_{12}^{(34)} S - \frac{1}{2} \Delta J_{12} S - \frac{1}{4} \zeta S + H_{I-II}, \]
\[ b_{16} = \frac{1}{2} \Delta J_{12} S + \frac{1}{4} \zeta S + G_{I-II}, \]
\[ b_{55} = \frac{1}{4} \zeta S - 16\xi C_{2b} + 2(K_1 - K_2 s^2 - K_3 c^2)S c_z, \]  
\[ b_{56} = 4J_2 S - 2\Delta J_2 S - \zeta S + 48\xi C_{2b} + 2(K_2 + K_3)S c_z. \]  

(In the above tabulation we not have included dipolar interactions. These are easiest to include when we give the mode energies because these terms can then be combined via Eq. (87) with the pseudodipolar terms which we treated explicitly.)

In Table I we summarize the definitions of the various parameters and in Table II we give estimates of their numerical values.

**A. CuII’s Ordered**

1. **Without 1/S Renormalizations**

Here we evaluate the energies of the four low-frequency modes in the presence of CuII ordering without any 1/S renormalizations. In what follows we will work to an accuracy of about 1%. That is, the only corrections of relative order 1/J we will keep are those of order \( J_{12}/J \) or \( J_2/J \). Then, in the notation of Eqs. (34) and (37) the components of the large matrix \([A + \sigma B]_\sigma\) are

\[ U_{\sigma} = 8JS, \quad V_{\sigma} = 2\sqrt{2}\sigma J_{12} S, \quad W_{\sigma} = 8J_2 S. \]  

We neglect terms which are small compared to \( \alpha \) and obtain
\[
[A - B]_{\sigma=+1} = \begin{bmatrix}
4\Delta J_1S + \alpha + x_3 & -\sqrt{2}\alpha \\
-\sqrt{2}\alpha & 4\Delta J_2S + 2\alpha
\end{bmatrix}
\]

for the out-of-plane sector, where \(x_3 = 2J_3S(1 - c_z)\), and

\[
[A + B]_{\sigma=-1} = \begin{bmatrix}
\zeta S + \alpha + 64C_2\tau + x_3 & \sqrt{2}(\alpha - \zeta S) \\
\sqrt{2}(\alpha - \zeta S) & 2(\zeta S + \alpha) - 64\xi C_2 + 4\Delta K_{\text{eff}}S(1 + c_z)
\end{bmatrix}
\]

for the in-plane sector, where \(\Delta K_{\text{eff}}\) was defined by Eq. (87).

From Eq. (36) we get the higher frequency modes as

\[
(\omega_+^\infty)^2 = 8JS(4\Delta J_1S + \alpha + x_3) - 8J_{12}S\alpha + \frac{16J_2S\alpha^2}{\alpha + 4\Delta J_1S + x_3}
\]

(97a)

\[
(\omega_-^\infty)^2 = 8JS(\alpha + x_3) + 8S\alpha(-J_{12} + 2J_2) - \frac{16J_2S\alpha x_3}{\alpha + x_3},
\]

(97b)

and the lower frequency modes as

\[
(\omega_+^\infty)^2 = 64JJ_2S^2\left([4\Delta J_1S + \alpha + x_3][4\Delta J_2S + 2\alpha] - 2\alpha^2\right)/(\omega_+^\infty)^2
\]

(97c)

\[
(\omega_-^\infty)^2 = 64JJ_2S^2\left([\zeta S + \alpha + 64C_2\tau + x_3][2\zeta S + 2\alpha - 64\xi C_2 + 4\Delta K_{\text{eff}}S(1 + c_z)]
\]

\[
-2(\alpha - \zeta S)^2)/\omega_-^\infty)^2
\]

\[
\approx \left(\frac{64JJ_2S^2\alpha}{(\omega_-^\infty)^2}\right)[2x_3 + 64(2\tau - \xi)C_2 + 8\zeta S + 4\Delta K_{\text{eff}}S(1 + c_z)]
\]

(97d)

In obtaining the above results we replaced \(UV - V^2\) by \(UV\) with an error of order 1%. To obtain the last line of Eq. (97d) we assumed that \(\alpha\) dominates the other perturbations.

As we have already seen, quantum fluctuations of the frustrated CuI - CuII interactions cause \(\omega_+^\infty\) to be nonzero even if the exchange interactions are isotropic. When we introduce easy plane anisotropy (by making \(\Delta J_1\) and/or \(\Delta J_2\) nonzero) we introduce a gap into \(\omega_-^\infty\), but \(\omega_-^\infty\) has no gap yet, because without in-plane anisotropy a global rotation of spins within the easy plane costs no energy. The lowest mode develops a gap when we introduce the in-plane anisotropy and take account of quantum fluctuations. One might imagine that the strongest such anisotropy, namely that in \(J\) (scaled by the parameter \(\delta J_1\)) would dominate in \(\omega_-^\infty\). This effect is incorporated in the term proportional to \(\tau = \delta J_1^2/J\), and indeed when the CuII’s are not ordered this term is the only one which contributes at \(q_z = 0\). However, when the CuII’s are ordered, the situation is
different. Notice that this factor has no factor of $S$ and more importantly, it is accompanied by the small numerical factor $C_2 \approx 0.01$. These observations remind us that this effect is another fluctuation effect. Within harmonic theory or mean-field theory the anisotropy of these CuI-CuI in-plane interactions averages to zero. In contrast, the weaker in-plane interaction between CuI's and CuII's (scaled by $\zeta \equiv (\delta J_{12})^2/J$) appears already in mean-field theory. Thus, this term, which is proportional to $S$, has no factor analogous to $C_2$ and it would dominate the term proportional to $\tau$ except for the fact (see next section) that its renormalization factor $Z_\zeta$ is quite small. However, when the CuII's are ordered, the interplanar CuII-CuII dipolar interactions contained in $\Delta K_{\text{eff}}$ are dominant, and lead to the dramatic increase in the effective four-fold anisotropy observed at low temperatures. The isotropic interplanar nearest neighbor CuI-CuII are frustrated. The anisotropic CuI-CuII interlayer interactions (as embodied by the constants $G$ and $H$) have only a negligible effect on the mode energies.

2. $1/S$ Renormalizations

In this subsection we summarize how we incorporate the various renormalizations due to spin-wave interactions. We believe that the correct procedure is to calculate the mode energies correctly at first order in $1/S$ and then set $S = 1/2$. Following this prescription we thereby obtain the following results

\[
\begin{align*}
(\omega^+)^2 &= 8JS \left[ \alpha + 4\Delta J_1 S Z_g^2 + x_3 Z_3^2 \right] - 8J_{12} S \alpha + \frac{16 J_2 S \alpha^2}{\alpha + 4\Delta J_1 S Z_g^2 + x_3 Z_3^2} \quad (98a) \\
(\omega^-)^2 &= 8JS \left[ \alpha + x_3 Z_3^2 \right] - 8J_{12} S \alpha + \frac{16 J_2 S \alpha^2}{\alpha + x_3 Z_3^2} \quad (98b) \\
(\omega_+^\perp)^2 &= 64JJ_2 S^2 \left[ 4\Delta J_1 S Z_g^2 + \alpha + x_3 Z_3^2 \right] \left[ 4\Delta J_2 S Z_g^2 + 2\alpha \right] - 2\alpha^2 \left/ (\omega^+)^2 \right. \quad (98c) \\
(\omega_-^\perp)^2 &= 64JJ_2 S^2 \left[ \zeta S + \alpha + 64C_2 \tau + Z_3^2 x_3 \right] \times \left[ 2\zeta S + 2\alpha - 64\xi C_2 + 4\Delta K_{\text{eff}} S(1 + c_2) \right] - 2\left[ \alpha - \zeta S \right]^2 \left/ (\omega^-)^2 \right. \quad (98d)
\end{align*}
\]

Here we noted that spins not in the same plane are essentially uncorrelated and hence we have
\[ J_3 \rightarrow \tilde{Z}_3 J_3, \quad \Delta K_{\text{eff}} \rightarrow \tilde{Z}_3 \Delta K_{\text{eff}}, \] (99)

where Eq. (40) gives \( \tilde{Z}_3 \approx 1 - 0.2/S \rightarrow 0.6 \). But since \( J_3 \) and \( \Delta K_{\text{eff}} \) always enter the mode energies in combination with an isotropic exchange constant, we associate with them the renormalizations

\[ J_3 \rightarrow Z_3^2 J_3, \quad \Delta K_{\text{eff}} \rightarrow Z_3^2 \Delta K_{\text{eff}}, \] (100)

where \( Z_3^2 = \tilde{Z}_3 Z_c \). Also, we will determine \( Z_\zeta \) by comparison, in Eq. (108) below, with the phenomenological treatment\(^4\) of the statics. For convenience we summarize in Table III the renormalizations of the various interactions which follow from our treatment to order \( 1/S \).

### B. CuII’s Disordered

To get the energies of the spin-wave modes when the CuII’s are disordered one sets \( J_{12} = J_2 = 0 \) (i.e., modes \( \omega_+ \) and \( \omega_- \) no longer exist as elementary excitations) and \( \alpha = 0 \), in which case we get

\[
(\omega_+)^2 = 8JS \left[ 4\Delta J_1 S Z_9^2 + 2J_3 S Z_3^2 (1 - c_z) \right], \tag{101a}
\]

\[
(\omega_-)^2 = 8JS \left[ 64\tau C_2 + 2J_3 S Z_3^2 (1 - c_z) \right]. \tag{101b}
\]

Note that in Eq. (97b) we had dropped a term representing the four-fold anisotropy which is proportional to \( \tau \), because such a term is negligible in comparison to \( \alpha \). Here, with \( \alpha \) not present, we restore this term in \( \omega_- \). Note also that the higher energy mode is the one which has fluctuations out of the plane (as indicated by the dependence on \( \Delta J_1 \)) and at zero wave vector is of the expected form, \( \omega^2 = 2H_E H_A \), with the exchange field \( H_E = 4JS \) and the anisotropy field \( H_A = 4\Delta J_1 S \). The energy of this out-of-plane gap is about 5 meV in many lamellar copper oxide antiferromagnets.\(^2\) The lower energy mode involves motion of the spins within the plane and would have no gap at zero wave vector except for the appearance of a small effective four-fold anisotropy, which was obtained previously\(^13\) from phenomenological considerations. The same result for the gap, namely \( \omega = 16\delta J_1 \sqrt{2C_2 S} \approx 1.6\delta J_1 \), is obtained from the microscopic calculation given in Appendix C and also in Ref. 24.
C. Comparison of Static and Dynamic Theories

Here we briefly compare our results with those of a mean-field treatment of the statics.\textsuperscript{4} In that calculation the four-fold anisotropy is included phenomenologically and the anisotropic CuI - CuII interactions are included even when the CuII sublattice is not antiferromagnetically ordered. When the CuII sublattice is ordered, the static treatment assumes that the Shender mechanism is strong enough that all spins are essentially collinear. So the dynamics of the Goldstone mode should involve the static response coefficients, although spin-wave hydrodynamics\textsuperscript{28} rigorously applies only in the limit of zero frequency.

Since the statics treat the four-fold anisotropy phenomenologically, as did Yildirim \textit{et al.},\textsuperscript{13} we identify their four-fold anisotropy constant $K$, which scales the anisotropy energy per CuI spin, from

$$E = -\frac{1}{2}K \cos(4\theta),$$

because there are two CuI's per unit cell. Also $\theta$ is the angle of the magnetic moment with respect to the easy, $(1,0,0)$, axis. In Ref. 13 the energy per CuI spin is (in the present notation)

$$E = 32C_2\tau S(S_x^2S_y^2/S^4).$$

So we make the identification $K = 8C_2\tau S$, or, if we include the effects of the CuII's,

$$K = 4C_2(2\tau - \xi)S.$$

We start by comparing the results of the two approaches when the CuII's are disordered. There the spin-wave calculation completely ignores the presence of the CuII's, whereas in the statics the CuII's are characterized by their susceptibility in the pseudo-dipolar field caused by the small in-plane anisotropy of the CuI - CuII interactions. In the statics for temperatures far below the ordering temperature for the CuI sublattice (but still with the CuII’s disordered) one has the effective fourth-order anisotropy constant $k_{\text{stat}}$ from the statics as

$$k_{\text{stat}} = 2K + 8M_0^2J_{12}^2\chi_I[1 - 8\chi_{II}^2J_{12}^2]^{-1},$$

where we introduce the Cu spin susceptibilities, $\chi_I \approx 0.53/(8J)$, $\chi_{II} \approx 0.53/(8J_2)$, and (in the present notation)
\[ M_0 = 4\delta J_{12}\langle S \rangle \chi_{II}, \]  

where \( \langle \cdot \rangle \) denotes a thermal average. If one takes \( \delta J_{12} = 0.025 \text{ meV} \), then \( M_0 = 2 \times 10^{-4} \). Then the second term on the right-hand side of Eq. (105) is about \( 6 \times 10^{-8} \text{ meV} \), compared with \( 2K \) which was found\(^4\) to be \( 2 \times 10^{-6} \text{ meV} \). So this correction (due to paramagnetic CuII's) which is absent from our spin-wave analysis is negligible.

When the CuII's are well ordered, Ref. 4 gives approximately

\[ k_{\text{stat}} = 2K + 8(\delta J_{12})^2\langle S \rangle^2\frac{0.53/(8J_2)}{0.53} \]  

Using Eq. (104) as the identification of \( K \), we see from Eq. (98d) that the mode energy involves the combination (for \( \xi \ll \tau \) and \( \zeta S \ll \alpha \) which we identify to be the effective value of \( k \) from the dynamics, \( k_{\text{dyn}} \), where

\[ k_{\text{dyn}} = 8(2\tau - \xi)C_2S + \zeta S^2Z_\zeta + \Delta K_{\text{eff}}S^2Z_\zeta = 2K + (\delta J_{12})^2S^2Z_\zeta/J_2 + \Delta K_{\text{eff}}S^2Z_3^2 \]  

We see that the term \((0.53)\langle S \rangle^2\) in the statics appears as \( S^2Z_\zeta \) in the spin-wave dynamics. With an appropriate renormalization \( Z_\zeta \approx 0.19 \), these two terms are the same. Thus, as far as the intralayer interactions are concerned the comparison between statics and dynamics indicates that these terms are correctly treated. We also see that the treatment of the statics did not include the interplanar anisotropic interaction, \( \Delta K_{\text{eff}} \). As we shall see, this term gives an important contribution to the mode \( \omega \leq \), so it should be included in a reanalysis of the statics. In terms of the constant \( k_{\text{dyn}} \) we may write Eq. (98d) as

\[ (\omega \leq)^2(q = 0) = 64J_2k_{\text{dyn}}\left( \frac{J}{J - J_{12} + 2J_2} \right) \]  

Thus we conclude that except for the fact that the statics ignored the interplanar anisotropic CuII-CuII interactions, the two theoretical approaches are compatible with one another. In the next section will show that the experimental results from static and dynamic measurements are also consistent with one another.

**D. Comparison to Experiments**

The comparison between the present theory and experiments has been described briefly in several previous publications\(^3,5,16\) Since a more detailed comparison is given in I, we will
simply summarize the comparison of the theoretical and experimental results. First one has
the estimate for $J$ which is nearly the same for all cuprates. This estimate has been refined by
Kim, who gives $J = 130$ meV. The value $J_2 = 10.5$ meV has been accurately determined by
comparing the experimental dispersion with respect to in-plane wave vector of CuII spin waves
to various theoretical treatments which take account of spin-wave interactions.

Now we discuss the analysis of the magnon gaps at zero wave vector where values are listed
in Table IV. We first fit the observed in-plane gap when the CuII's are disordered. Equation
(101b) yields $\omega_- = 16\sqrt{2C_2SgJ_1}$ and with $\omega_- = 0.066$ meV, we get $|\delta J_1| = |J_\parallel - J_\perp|/2 = 0.042$
meV, a value which is about twice the theoretical estimates. Using Eqs. (104) and (105) this
corresponds to $k = 16C_2S\tau = 16(0.01)(0.5)(0.042)/130 = 1 \times 10^{-6}$ meV, compared to the
value deduced from the statics, $k = 2 \times 10^{-6}$ meV, for $70K \leq T \leq 120$K. At low temperature
($T = 1.4K$), where the CuII's are well ordered, the statics gives $k = 25 \times 10^{-6}$ meV. From Eq.
(109) with $\omega_- = 0.15$ meV, we get $k_{dyn} = 41 \times 10^{-6}$ meV. These results are listed in Table V,
where we see only a qualitative consistency between the interpretation of the static and dynamic
experiments. It is possible that the quantum renormalizations (which affect the determination
of $k$ from the observed mode energy) are not quite correct. Also, the interpretation of the statics
within which the CuII-CuII interplanar anisotropy is subsumed into the four-fold anisotropy
constant, $k$, is not strictly correct. If we fix $\delta J_1$ to fit the value of $\omega_- \leq$ at $T = 100$K and assume
that the interplanar CuII-CuII interactions result from the actual dipole-dipole interactions,
then the temperature dependence of $k$ results from the last term in Eq. (108). With only
dipolar (i.e., no pseudo dipolar) interactions, Eqs. (87) and (85) give $\Delta K_{eff} = 242 \times 10^{-6}$ meV,
so that $\Delta K_{eff}S^2 \Sigma_3^2 = 42 \times 10^{-6}$ meV, from which $k_{dyn} = 45 \times 10^{-6}$ meV. From Table V it is clear
that the experimentally deduced temperature dependence of $k$ is reasonably well accounted for
by the intraplanar dipolar interactions, especially if one increases $\Delta K$ by assuming it to have
a small pseudodipolar component.

Now we consider the higher energy modes. Fitting to the observed energy $\omega_+ = 5.5$ meV
of the out-of-plane gap when the CuII's are disordered to Eq. (101a) (with $Z_g = 0.6$) we obtain
the value of $\Delta J_1 = 0.081$ meV. As was the case for $\delta J_1$, this result is also about twice the
theoretical estimates for a simple CuO plane. Given the values of these parameters, both
higher-energy modes at low temperature involve only the one additional parameter, \( \alpha \). If we determine \( \alpha \) from \( \omega_+^\infty \) we get \( \alpha = 0.14 \text{ meV} \), whereas if we determine \( \alpha \) from \( \omega_+^\approx \) we get \( \alpha = 0.13 \text{ meV} \). These two values agree perfectly with one another and their average coincides with the theoretical evaluation of Appendix B that \( \alpha = 0.13 \text{ meV} \). Clearly these agreements strongly support our interpretation of the role of fluctuations embodied by the parameter \( \alpha \). Note that a biquadratic interaction between two spin 1/2’s can be subsumed into a ordinary Heisenberg exchange interaction. Therefore biquadratic can not contribute to \( \Delta K_{\text{eff}} \).

Finally we consider the lower energy out-of-plane mode in the zero temperature limit. The AFMR data\(^{15,16} \) gives \( \omega_-^\approx = 1.75 \text{ meV} \), more accurate than, but entirely consistent with, the data of Ref. 7. Fitting this to the expression in Eq. (98c) with \( \Delta J_2 = 0 \) gives \( \omega_-^\approx = 1.70 \text{ meV} \). If we fix \( \Delta J_2 \) to fit the experimental value of this gap, we get \( \Delta J_2 = 0.008 \pm 0.008 \text{ meV} \). We attribute a large uncertainty to \( \Delta J_2 \) because its value changes significantly if \( \Delta J_1 \) or \( \alpha \) is slightly modified. To get the same relative out-of-plane anisotropy, \( \Delta J/J \), for the CuII-CuII exchange as for the CuI-CuI exchange would require \( \Delta J_1 = 0.008 \text{ meV} \).

**VII. DYNAMIC STRUCTURE FACTOR**

The cross section, \( \sigma(q, \omega) \), for inelastic neutron scattering from magnetic ions is proportional to the dynamic structure factor \( S^{\alpha \beta}(q, \omega) \) which in turn is related to the spin-spin correlation function. We have

\[
\sigma(q, \omega) \propto \sum_{\alpha \beta} (\delta_{\alpha, \beta} - q_{\alpha} q_{\beta}) S^{\alpha \beta}(q, \omega) . \tag{110}
\]

According to the fluctuation-dissipation theorem, we may write

\[
S^{\alpha \beta}(q, \omega) = \frac{1}{\pi} n(\omega) \text{Im} \chi^{\alpha \beta}(q, \omega - i0^+) , \tag{111}
\]

where \( n(\omega) = [e^{\hbar \omega/(kT)} - 1]^{-1} \) and, in the usual notation,\(^{32} \) the \( A - B \) Green’s function is defined as

\[
\langle \langle A; B \rangle \rangle_{\omega} = \sum_{m,n} p_n \left[ \frac{\langle n | A | m \rangle \langle m | B | n \rangle}{\omega - E_m + E_n} - \frac{\langle n | B | m \rangle \langle m | A | n \rangle}{\omega + E_m - E_n} \right] , \tag{112}
\]

37
where \(|n\rangle\) and \(|m\rangle\) are exact eigenstates with respective energies \(E_n\) and \(E_m\) and \(p_n\) is the Boltzmann weight of the state \(|n\rangle\). Then \(\chi\), the dynamic susceptibility, is written as the Green’s function

\[ \chi^{\alpha\beta}(q, \omega) = \langle \langle S^\alpha(q); S^\beta(-q) \rangle \rangle_\omega. \]  

(113)

We construct the dynamic susceptibility by writing the spin operators in terms of boson operators at leading order in \(1/S\):

\[
S^+(q) = \sqrt{2S} \left( a(q) + b^\dagger(-q) + c^\dagger(-q) + d(q) + e^\dagger(-q) + f(q) \right)
\]

\[
S^-(q) = \sqrt{2S} \left( a^\dagger(q) + b(-q) + c(-q) + d^\dagger(q) + e(-q) + f^\dagger(q) \right).
\]

(114)

Thus we have

\[ S_\eta(q) = [S^+(q) + S^-(q)]/2 = \sqrt{S/2} \sum_m \left[ V_m(\eta) \xi_m(q) + V_m(\eta)^* \xi_m^\dagger(-q) \right], \]

(115)

and

\[ S^z(q) = -i[S^+(q) - S^-(q)]/2 = \sqrt{S/2} \sum_m \left[ V_m(z) \xi_m(q) + V_m(z)^* \xi_m^\dagger(-q) \right], \]

(116)

where the operators are labelled as in Eq. (15) and the transpose of the column vectors \(\mathbf{V}(\alpha)\) is

\[ \bar{V}(\eta) = (1, 1, 1, 1, 1), \quad \bar{V}(z) = i(1, -1, -1, 1, -1). \]

(117)

Thus we may write

\[ \chi^{\alpha\beta}(q, \omega) = \frac{i}{2} S \sum_{mn} \langle \langle [V_m(\alpha) \xi_m(q) + V_m(\alpha)^* \xi_m^\dagger(-q)] : [V_n(\beta) \xi_n(-q) + V_n(\beta)^* \xi_n^\dagger(q)] \rangle \rangle_\omega. \]

(118)

We may evaluate these response functions in terms of normal modes. Suppose we have found the unnormalized right eigenvectors of the dynamical matrix, Eq. (23). That is we have the column vectors \(\Phi_j\) which satisfy

\[ [\mathbf{A} + \mathbf{B}][\mathbf{A} - \mathbf{B}] \Phi_j = \omega_j^2 \Phi_j. \]

(119)

Then we make the identification that

\[ \Phi_j = \xi^\dagger_j(q) \xi_j(-q) \]

\[ \Phi_j = \xi^\dagger_j(-q) \xi_j(q) \]

\[ \Phi_j = \xi^\dagger_j(q) \xi_j(q) \]

\[ \Phi_j = \xi^\dagger_j(-q) \xi_j(-q) \]
\[ P_j - Q_j = x_j \Phi_j \, . \] (120)

We can arbitrarily fix the phase of the normal mode operators so that \( x_j \) is real positive. Then

\[ [A - B] x_j \Phi_j = [A - B] [P_j - Q_j] = \omega_j [P_j + Q_j] \, , \] (121)

or

\[ P_j + Q_j = (x_j/\omega_j) [A - B] \Phi_j \, , \] (122)

so that

\[
\begin{align*}
P_j &= \frac{x_j}{2} \left[ I + \omega_j^{-1} [A - B] \right] \Phi_j \\
Q_j &= \frac{x_j}{2} \left[ -I + \omega_j^{-1} [A - B] \right] \Phi_j \, .
\end{align*}
\] (123)

To use Eq. (18) we write

\[ P_j^\dagger P_j - Q_j^\dagger Q_j \equiv \frac{x_j^2}{\omega_j} \left( \Phi_j^\dagger [A - B] \Phi_j \right) \, , \] (124)

so that

\[ x_j^2 = \frac{\omega_j}{[\Phi_j^\dagger [A - B] \Phi_j]} \, . \] (125)

Then we write the susceptibilities as

\[
(2/S) \chi^{\alpha \beta}(q, \omega) = \sum_{m,n,r} \left( [V_m(\alpha) P_{m\tau}(q) + V_m(\alpha)^* Q_{m\tau}(q)] \\
\times [V_n(\beta)^* P_{n\tau}(q)^* + V_n(\beta)^* Q_{n\tau}(q)^*] \langle \tau_r(q); \tau_r^\dagger(q) \rangle \omega \\
+ \sum_{mnr} \left( [V_m(\alpha) Q_{mr}(q) + V_m(\alpha)^* P_{mr}(q)] \\
\times [V_n(\beta) P_{nr}(q)^* + V_n(\beta) Q_{nr}(q)^*] \langle \tau_r^\dagger(q); \tau_r(q) \rangle \omega \\
= \sum_r \left\{ \left[ (\tilde{V}(\alpha) P_r) + (\tilde{V}(\alpha)^* Q_r) \right] \left[ (\tilde{V}(\beta) P_r) + (\tilde{V}(\beta)^* Q_r) \right] \right\} \omega - \omega_r(q) \right)^{-1} \\
+ \left[ (\tilde{V}(\alpha) Q_r) + (\tilde{V}(\alpha)^* P_r) \right] \left[ (\tilde{V}(\beta) Q_r) + (\tilde{V}(\beta)^* P_r) \right] \right\} \omega + \omega_r(q) \right)^{-1} \\
= \sum_r \left[ \frac{J^{\alpha \beta}_r(q)}{\omega - \omega_r(q)} + \frac{I^{\alpha \beta}_r(q)}{\omega + \omega_r(q)} \right] \, , \]
\] (126)
where we left the argument \( q \) implicit in several places. We will refer to \( I \) and \( J \) as 'intensities', although to get inelastic neutron scattering cross-sections one needs to include several other factors. At low temperature we only need

\[
I_{r}^{\alpha \beta}(q) = x_{r}^{2} \left[ \delta_{\alpha, z} \left( V(z)^{\dagger} \Phi_{r}(q) \right) + \omega_{r}(q)^{-1} \delta_{\alpha, \eta} \left( V(\eta)^{\dagger}[A - B] \Phi_{r}(q) \right) \right] \\
\times \left[ \delta_{\beta, z} \left( V(z)^{\dagger} \Phi_{r}(q) \right) + \omega_{r}(q)^{-1} \delta_{\beta, \eta} \left( V(\eta)^{\dagger}[A - B] \Phi_{r}(q) \right) \right]^*. \tag{127}
\]

In writing this result we used the fact that \( V(\eta) \) is real and \( V(z) \) is imaginary. From now on, we specialize to the case of wave vectors of the form \( q = G + q_{z} \hat{z} \). In that case \( I_{r}^{\eta \eta} + I_{r}^{z z} \) vanishes and

\[
I_{r}^{zz} = \frac{\left| \left( V(z)^{\dagger} \Phi_{r}(q) \right)^{2} \omega_{r}(q) \right|}{\Phi_{r}(q)^{\dagger}[A - B] \Phi_{r}(q)} \tag{128a}
\]

\[
I_{r}^{\eta \eta} = \frac{\left| \left( V(\eta)^{\dagger}[A - B] \Phi_{r}(q) \right)^{2} \right.}{\omega_{r}(q) \left( \Phi_{r}(q)^{\dagger}[A - B] \Phi_{r}(q) \right)}. \tag{128b}
\]

The above results are useful for the out-of-plane (\( \sigma = +1 \)) modes in which case \([A - B] \) is the small matrix. Alternatively, for in-plane (\( \sigma = -1 \)) modes when \( A + B \) is the small matrix the following forms are useful:

\[
I_{r}^{zz} = \frac{\left| \left( V(z)^{\dagger}[A + B] \Psi_{r}(q) \right)^{2} \right|}{\omega_{r}(q) \left( \Psi_{r}(q)^{\dagger}[A + B] \Psi_{r}(q) \right)} \tag{129a}
\]

\[
I_{r}^{\eta \eta} = \frac{\left| \left( V(\eta)^{\dagger} \Psi_{r}(q) \right)^{2} \omega_{r}(q) \right|}{\left( \Psi_{r}(q)^{\dagger}[A + B] \Psi_{r}(q) \right)}. \tag{129b}
\]

For high symmetry directions of the wave vector, the matrices \( A \) and \( B \) may be brought into block diagonal form by a unitary transformation \( U \). In that case we may apply the above formulas in terms of the transformed quantities indicated by primes:
\[ A' \equiv U^\dagger A U, \quad B' \equiv U^\dagger B U, \]
\[ \Phi'_r \equiv U^\dagger \Phi_r , \quad \Psi'_r \equiv U^\dagger \Psi_r , \quad V'(\alpha) \equiv U^\dagger V(\alpha) . \]  \hspace{1cm} (130)

For wave vectors which are equal modulo a reciprocal lattice vector, the corresponding quantities, \( A', B', \Phi', \) and \( \Psi' \) are equal. However, the intensities at such equivalent points will differ because \( U \), and hence \( V' \), depend specifically on the zone of the wave vector. This can be seen explicitly in Appendix A where we obtain the results summarized in Tables VI and VII. Note that the \( \sigma = +1 \) sector does have intensity mainly in \( I_{zz} \) in confirmation of our identification of this as the out-of-plane sector. Similarly, the \( \sigma = -1 \) sector has its intensity mainly in \( I_{yy} \) as expected for in-plane modes. These identifications are also consistent with the fact that the \( \sigma = +1 \) modes depend on the out-of-plane anisotropies scaled by the \( \Delta J' \)'s, whereas the \( \sigma = -1 \) modes do not involve these quantities.

VIII. CONCLUSIONS

Here we briefly summarize the significant conclusions from this work.

- 1. The degeneracy, present within mean-field theory, in which the CuII sublattice spins can be globally rotated with respect to the CuI spins is removed by quantum fluctuations which cause the sublattice magnetizations to be collinear, as first indicated by Shender.\(^8\)

- 2. A degeneracy present within mean-field and linear spin-wave theories, in which the magnetization can be globally rotated through an arbitrary angle within the easy plane is similarly removed by quantum fluctuations, as first proposed in Ref. 13.

- 3. These fluctuation effects, in addition to selecting the ground state from among the classically degenerate configurations, also give rise to nonzero energies of the corresponding spin-wave excitations. The most dramatic evidences of this phenomenon are the striking increases of the out-of-plane gap energy from 5 to 10 meV and that of the in-plane gap from zero to 9 meV when the CuII sublattice evolves from disorder to order.

- 4. The experimental results of inelastic neutron scattering for the lowest energy gaps are broadly consistent with the effective four-fold anisotropy previously obtained from the statics experiments.\(^4\) More precise agreement may depend on more accurate understanding of the
various renormalizations due to quantum and thermal fluctuations.

- 5. Our improved theoretical treatment which now includes the interlayer dipolar interactions resolves the mystery surrounding the dramatic increase (first found in the statics\(^4\)) in the effective four-fold anisotropy as the temperature is reduced into the regime where the CuII's order. In fact the dipolar interlayer interactions between the CuII's dominates the effective four-fold anisotropy when the CuII's develop long range order.

- 6. Recent AFMR results\(^6\) lead to an identification of the small in-plane anisotropies and qualitatively confirm previous theoretical estimates of the exchange anisotropy induced by spin-orbit interactions.\(^13,14\)

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APPENDIX A: INTENSITY CALCULATIONS

In this Appendix we evaluate the intensities for which formulas are given in Sec. V. We first give the unitary transformation which brings the matrices $A$ and $B$ into block diagonal form. We do this for wave vectors $q = G + q_z \hat{z}$, where

$$G = 2\pi \left[ \frac{H_\hat{x}}{a} + \frac{K_\hat{y}}{a} + \frac{L_\hat{z}}{c} \right] \quad \text{(A1)}$$

where $H$ and $K$ are either both half integral or both integral and $H + K + L$ is an even integer. Then

$$U = \frac{1}{2} \begin{pmatrix} \sqrt{2} & 0 & 1 & 0 & 1 & 0 \\ 0 & \sqrt{2} & (-1)^{H+K} & 0 & -(-1)^{H+K} & 0 \\ 0 & -(-1)^{2H} \sqrt{2} & (-1)^{H-K} & 0 & -(-1)^{H-K} & 0 \\ -(-1)^{2H} \sqrt{2} & 0 & (-1)^{2H} & 0 & (-1)^{2H} & 0 \\ 0 & 0 & 0 & (i)^{2H} \sqrt{2} & 0 & (i)^{2H} \sqrt{2} \\ 0 & 0 & 0 & (i)^{2H} \sqrt{2} & 0 & -(i)^{2H} \sqrt{2} \end{pmatrix} . \quad \text{(A2)}$$

The first two columns are the high frequency CuI optical modes. Columns #3 and 4 are the $\sigma = 1$ out-of-plane modes and columns #5 and 6 are the $\sigma = -1$ in-plane modes. The following results hold for all wavevectors of the form $q = G + q_z \hat{z}$.

1. Out-of-Plane Modes

For the out-of-plane sector we have (for dominant $J$)

$$A' - B' = \begin{pmatrix} x_3 + 4\Delta J_1 S + \frac{1}{2}\alpha & \alpha/\sqrt{2} \\ \alpha/\sqrt{2} & 4\Delta J_2 S + \alpha \end{pmatrix} , \quad A' + B' = \begin{pmatrix} 8JS & \sqrt{2}J_{12}S \\ \sqrt{2}J_{12}S & 8J_2S \end{pmatrix} \quad \text{(A3)}$$

independent of $G$, where $x_3 = 2J_3 S [1 - \cos(q_z/2)]$. Note that $q_z$ is measured relative to the reciprocal lattice vector in question. We now tabulate the right eigenvectors of the block matrices $M_{++} = [A' + B'] [A' - B']$ associated with the eigenvalues (the squares of the mode energies) $\omega^2_r$. We have

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\( \Phi_+ = [1,0] \), \( (\omega_+^2 = (8JS)(x_3 + 4\Delta J_1S + \frac{1}{2}\alpha) \),

\( \Phi_- = [-\alpha/\sqrt{2}, x_3 + 4\Delta J_1S + \frac{1}{2}\alpha], \ (\omega_-^2 = (8J_2S) \left[ 4\Delta J_2S + \alpha - \frac{1}{2} \frac{\alpha^2}{x_3 + 4\Delta J_1S + \frac{1}{2}\alpha} \right] \). (A4)

Also we find that

\[
V(z)' = \begin{bmatrix} 1 - (-1)^{H+K} \\ 0 \end{bmatrix} \quad V(\eta)' = \begin{bmatrix} 1 + (-1)^{H+K} \\ \sqrt{2}(-1)^H \end{bmatrix} \text{ for integer } H \quad (A5a)
\]

\[
V(z)' = \begin{bmatrix} 0 \\ i^{2H}\sqrt{2} \end{bmatrix} \quad V(\eta)' = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ for half - integer } H \quad (A5b)
\]

Note that the vectors \( V(\alpha)' \) depend on \( G \). Substituting these evaluations into Eq. (128) we obtain the results for the intensities in Tables VI and VII for the out-of-plane \( (\sigma = +1) \) modes.

### 2. In-Plane Modes

For the in-plane sector we have (for dominant \( J \))

\[
A' + B' = \begin{pmatrix} x_3 + \zeta S + \frac{1}{2} \alpha \\ \sqrt{2}(\zeta S - \frac{1}{2} \alpha) \\ \sqrt{2}(\zeta S - \frac{1}{2} \alpha) \sqrt{2} \zeta S + \alpha \end{pmatrix}, \quad A' - B' = \begin{pmatrix} 8JS \\ \sqrt{2}J_2S \zeta S \zeta S \sqrt{2} \zeta S + \alpha \end{pmatrix}, \quad (A6)
\]

and we now tabulate the right eigenvectors of the block matrices \( M_{\pm} = [A' - B'][A' + B'] \) associated with the eigenvalues (the squares of the mode energies) \( \omega_i^2 \). For dominant \( J \) we have the approximate results

\( \Phi_+ = [1,0] \), \( (\omega_+^2 = (8JS)(x_3 + \zeta S + \frac{1}{2} \alpha) \),

\( \Phi_- = [-\sqrt{2}(\zeta S - \frac{1}{2} \alpha), x_3 + \zeta S + \frac{1}{2} \alpha], \ (\omega_-^2 = (8J_2S) \left[ 2\zeta S + \alpha - 2 \frac{\zeta S - \frac{1}{2} \alpha)^2}{(x_3 + \zeta S + \frac{1}{2} \alpha)} \right] \) (A7)

and

\[
V(z)' = \begin{bmatrix} 1 + (-1)^{H+K} \\ -\sqrt{2}(-1)^H \end{bmatrix} \quad V(\eta)' = \begin{bmatrix} 1 - (-1)^{H+K} \\ 0 \end{bmatrix} \text{ for integer } H \quad (A8a)
\]

\[
V(z)' = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad V(\eta)' = \begin{bmatrix} 0 \\ -i^{2H}\sqrt{2} \end{bmatrix} \text{ for half - integer } H . \quad (A8b)
\]

As before, only the vectors \( V(\alpha)' \) depend on \( G \). Substituting these evaluations into Eq. (129) we obtain the results for the intensities in Tables VI and VII for the in-plane \( (\sigma = -1) \) modes.
APPENDIX B: SHENDER PARAMETERS

In this Appendix we evaluate the averages

\[ A_1 = \langle a_m f_n^\dagger \rangle \]  \hspace{1cm} (B1a)
\[ A_2 = \langle a_m e_n \rangle \]  \hspace{1cm} (B1b)

where site \( n \) is a nearest neighbor of site \( m \). The above quantities can be calculated perturbatively in the frustrated coupling \( J_{12} \) between CuI's and CuII's. (See Fig. 1.)

1. \( A_1 \)

Thus

\[ A_1 = - \left\langle 0 \left| \frac{1}{\mathcal{E}} V_{I-II} \right| a_m f_n^\dagger \right\rangle - \left\langle 0 \left| a_m f_n^\dagger \frac{1}{\mathcal{E}} V_{I-II} \right| 0 \right\rangle \]  \hspace{1cm} (B2)

where \( \mathcal{E} \) is the unperturbed energy of the virtual state relative to the ground state. Here we invoke perturbation theory relative to decoupled CuI and CuII subsystems, and

\[
V_{I-II} = J_{12} S \left[ \sum_{i \in a, \delta} [a_i a_i + e_j^\dagger e_j + a_i e_j + a_i^\dagger e_i^\dagger] + \sum_{i \in n, \delta} [a_i^\dagger a_i + f_j^\dagger f_j + a_i^\dagger f_j + f_j^\dagger a_i] \right.
\]
\[
+ \sum_{i \in b, \delta} [b_i^\dagger b_i + c_j^\dagger c_j + b_i^\dagger e_j + e_j^\dagger b_i] + \sum_{i \in b, \delta} [b_i^\dagger b_i + f_j^\dagger f_j + b_i^\dagger f_j + f_j^\dagger b_i] \]
\[
+ \sum_{i \in c, \delta} [c_i^\dagger c_i + e_j^\dagger e_j + c_i^\dagger e_j + e_j^\dagger c_i] + \sum_{i \in c, \delta} [c_i^\dagger c_i + f_j^\dagger f_j + c_i^\dagger f_j + f_j^\dagger c_i] \]
\[
+ \sum_{i \in d, \delta} [d_i^\dagger d_i + e_j^\dagger e_j + d_i^\dagger e_j + e_j^\dagger d_i] + \sum_{i \in d, \delta} [d_i^\dagger d_i + f_j^\dagger f_j + d_i^\dagger f_j + f_j^\dagger d_i] \right]
\]  \hspace{1cm} (B3)

Only terms in \( V_{I-II} \) which have operators in both subsystems contribute. So, effectively

\[
V_{I-II} = J_{12} S \sum_{i, \delta} \left[ a_i e_j + a_i^\dagger e_j^\dagger + a_i^\dagger f_j + f_j^\dagger a_i + b_i^\dagger b_i + e_j^\dagger b_i + b_i^\dagger f_j + f_j^\dagger b_i \right]
\]
\[
+ c_i^\dagger e_j + e_j^\dagger c_i + c_i^\dagger f_j + f_j^\dagger c_i + d_i^\dagger e_j + e_j^\dagger d_i + f_j^\dagger f_j + f_j^\dagger d_i \right] ,
\]  \hspace{1cm} (B4)

where site \( j \) is the appropriate nearest neighbor of site \( i \). In fact, in Eq. (B2) we need to have only terms with \( f \) or \( e^\dagger \) and \( a^\dagger, d^\dagger, b, \) or \( c \). So we set

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\[ V_{I-II} = V_i \equiv J_{12} \sum_{i,\delta} \left[ a_i^\dagger e_j^\dagger + a_i^\dagger f_j^\dagger + b_i^\dagger b_j^\dagger + c_i^\dagger c_j^\dagger + d_i^\dagger e_j^\dagger + d_i^\dagger f_j^\dagger \right]. \]  

(B5)

Thus with \( n = m + \delta_{af} \) we have

\[
A_1 = -J_{12} \sum_{i\in a} \left\langle 0 \left| a_i^\dagger \frac{1}{\mathcal{E}} a_m^\dagger \right| 0 \right\rangle \left\langle 0 \left| [e_i^\dagger e_{i+\delta_{ae}} + f_i^\dagger f_{i+\delta_{af}}] f_{m+\delta_{af}}^\dagger \right| 0 \right\rangle \\
- J_{12} \sum_{i\in b} \left\langle 0 \left| b_i^\dagger \frac{1}{\mathcal{E}} a_m^\dagger \right| 0 \right\rangle \left\langle 0 \left| [e_i^\dagger e_{i+\delta_{ae}} + f_i^\dagger f_{i+\delta_{af}}] f_{m+\delta_{af}}^\dagger \right| 0 \right\rangle \\
- J_{12} \sum_{i\in c} \left\langle 0 \left| c_i^\dagger \frac{1}{\mathcal{E}} a_m^\dagger \right| 0 \right\rangle \left\langle 0 \left| [e_i^\dagger e_{i+\delta_{ae}} + f_i^\dagger f_{i+\delta_{af}}] f_{m+\delta_{af}}^\dagger \right| 0 \right\rangle \\
- J_{12} \sum_{i\in d} \left\langle 0 \left| d_i^\dagger \frac{1}{\mathcal{E}} a_m^\dagger \right| 0 \right\rangle \left\langle 0 \left| [e_i^\dagger e_{i+\delta_{ae}} + f_i^\dagger f_{i+\delta_{af}}] f_{m+\delta_{af}}^\dagger \right| 0 \right\rangle \\
- J_{12} \sum_{i\in a} \left\langle 0 \left| a_m^\dagger \frac{1}{\mathcal{E}} a_i^\dagger \right| 0 \right\rangle \left\langle 0 \left| f_{m+\delta_{af}}^\dagger [e_i^\dagger e_{i+\delta_{ae}} + f_i^\dagger f_{i+\delta_{af}}] \right| 0 \right\rangle \\
- J_{12} \sum_{i\in b} \left\langle 0 \left| a_m^\dagger \frac{1}{\mathcal{E}} b_i^\dagger \right| 0 \right\rangle \left\langle 0 \left| f_{m+\delta_{af}}^\dagger [e_i^\dagger e_{i+\delta_{ae}} + f_i^\dagger f_{i+\delta_{af}}] \right| 0 \right\rangle \\
- J_{12} \sum_{i\in c} \left\langle 0 \left| a_m^\dagger \frac{1}{\mathcal{E}} c_i^\dagger \right| 0 \right\rangle \left\langle 0 \left| f_{m+\delta_{af}}^\dagger [e_i^\dagger e_{i+\delta_{ae}} + f_i^\dagger f_{i+\delta_{af}}] \right| 0 \right\rangle \\
- J_{12} \sum_{i\in d} \left\langle 0 \left| a_m^\dagger \frac{1}{\mathcal{E}} d_i^\dagger \right| 0 \right\rangle \left\langle 0 \left| f_{m+\delta_{af}}^\dagger [e_i^\dagger e_{i+\delta_{ae}} + f_i^\dagger f_{i+\delta_{af}}] \right| 0 \right\rangle. 
\]

(B6)

Here we neglected the energy of the CuII modes in comparison to that of the CuI modes. Also we used the unusual notation that

\[ \delta_{st} = r_t - r_s. \]

(B7)

Then

\[
A_1 = -\frac{J_{12} S}{N_{uc}^2} \sum_{q,k} \sum_{i\in a} \left\langle 0 \left| a^\dagger (q) \frac{1}{\mathcal{E}} a(q) \right| 0 \right\rangle e^{i(q+k)\cdot r_{im}} \\
\times \left\langle 0 \left| [e^\dagger (k) e^{ik\delta_{ae}} + f(-k) e^{ik\delta_{af}}] f^\dagger (-k) e^{-ik\delta_{af}} \right| 0 \right\rangle \\
- \frac{J_{12} S}{N_{uc}^2} \sum_{q,k} \sum_{i\in b} \left\langle 0 \left| b(-q) \frac{1}{\mathcal{E}} a(q) \right| 0 \right\rangle e^{i(q+k)\cdot r_{im}} \\
\times \left\langle 0 \left| [e^\dagger (k) e^{ik\delta_{ae}} + f(-k) e^{ik\delta_{af}}] f^\dagger (-k) e^{-ik\delta_{af}} \right| 0 \right\rangle \\
- \frac{J_{12} S}{N_{uc}^2} \sum_{q,k} \sum_{i\in c} \left\langle 0 \left| c(-q) \frac{1}{\mathcal{E}} a(q) \right| 0 \right\rangle e^{i(q+k)\cdot r_{im}} \\
\times \left\langle 0 \left| [e^\dagger (k) e^{ik\delta_{ae}} + f(-k) e^{ik\delta_{af}}] f^\dagger (-k) e^{-ik\delta_{af}} \right| 0 \right\rangle \\
- \frac{J_{12} S}{N_{uc}^2} \sum_{q,k} \sum_{i\in d} \left\langle 0 \left| d^\dagger (q) \frac{1}{\mathcal{E}} a(q) \right| 0 \right\rangle e^{i(q+k)\cdot r_{im}} \\
\times \left\langle 0 \left| [e^\dagger (k) e^{ik\delta_{ae}} + f(-k) e^{ik\delta_{af}}] f^\dagger (-k) e^{-ik\delta_{af}} \right| 0 \right\rangle.
\]
\[
\times \langle 0 | e^{\dagger}(k)e^{ik\delta_{ae}} + f(-k)e^{ik\delta_{be}} | f^\dagger(-k)e^{-ik\delta_{ae}} | 0 \rangle \\
- \frac{J_{12}}{N_{uc}} \sum_{q,k} \sum_{i,a} \left< 0 \right| a(q) \left( \frac{1}{\xi} a^{\dagger}(q) \right) \left| 0 \right> e^{i(q+k)\cdot r_m} \\
\times \langle 0 | f^\dagger(-k)e^{-ik\delta_{ae}} \left[ e^{\dagger}(k)e^{ik\delta_{ae}} + f(-k)e^{ik\delta_{be}} \right] | 0 \rangle \\
- \frac{J_{12}}{N_{uc}} \sum_{q,k} \sum_{i,b} \left< 0 \right| a(q) \left( \frac{1}{\xi} b(-q) \right) \left| 0 \right> e^{i(q+k)\cdot r_m} \\
\times \langle 0 | f^\dagger(-k)e^{-ik\delta_{be}} \left[ e^{\dagger}(k)e^{ik\delta_{be}} + f(-k)e^{ik\delta_{de}} \right] | 0 \rangle \\
- \frac{J_{12}}{N_{uc}} \sum_{q,k} \sum_{i,c} \left< 0 \right| a(q) \left( \frac{1}{\xi} c(-q) \right) \left| 0 \right> e^{i(q+k)\cdot r_m} \\
\times \langle 0 | f^\dagger(-k)e^{-ik\delta_{de}} \left[ e^{\dagger}(k)e^{ik\delta_{de}} + f(-k)e^{ik\delta_{de}} \right] | 0 \rangle .
\]

Doing the sum over \( i \) we get

\[
A_1 = - \frac{J_{12}}{N_{uc}} \sum_q \left< 0 \right| a^{\dagger}(q) \left( \frac{1}{\xi} a(q) \right) \left| 0 \right> \langle 0 | e^{\dagger}(-q)e^{-iq\delta_{ae}} + f(q)e^{-iq\delta_{ae}} | f^\dagger(q)e^{iq\delta_{ae}} | 0 \rangle \\
- \frac{J_{12}}{N_{uc}} \sum_q \left< 0 \right| b(-q) \left( \frac{1}{\xi} a(q) \right) \left| 0 \right> \langle 0 | e^{\dagger}(-q)e^{-iq\delta_{be}} + f(q)e^{-iq\delta_{be}} | f^\dagger(q)e^{iq\delta_{be}} | 0 \rangle \\
- \frac{J_{12}}{N_{uc}} \sum_q \left< 0 \right| c(-q) \left( \frac{1}{\xi} a(q) \right) \left| 0 \right> \langle 0 | e^{\dagger}(-q)e^{-iq\delta_{de}} + f(q)e^{-iq\delta_{de}} | f^\dagger(q)e^{iq\delta_{de}} | 0 \rangle \\
- \frac{J_{12}}{N_{uc}} \sum_q \left< 0 \right| d^\dagger(q) \left( \frac{1}{\xi} a(q) \right) \left| 0 \right> \langle 0 | e^{\dagger}(-q)e^{-iq\delta_{de}} + f(q)e^{-iq\delta_{de}} | f^\dagger(q)e^{iq\delta_{de}} | 0 \rangle \\
- \frac{J_{12}}{N_{uc}} \sum_q \left< 0 \right| \left( e^{\dagger}(-q)e^{-iq\delta_{ae}} + f(q)e^{-iq\delta_{ae}} \right) f^\dagger(q)e^{iq\delta_{ae}} | 0 \rangle \\
- \frac{J_{12}}{N_{uc}} \sum_q \left< 0 \right| b(-q) \left( \frac{1}{\xi} a(q) \right) \left| 0 \right> \langle 0 | f^\dagger(q)e^{iq\delta_{ae}} \left[ e^{\dagger}(-q)e^{-iq\delta_{ae}} + f(q)e^{-iq\delta_{ae}} \right] | 0 \rangle \\
- \frac{J_{12}}{N_{uc}} \sum_q \left< 0 \right| a(q) \left( \frac{1}{\xi} b(-q) \right) \left| 0 \right> \langle 0 | f^\dagger(q)e^{iq\delta_{be}} \left[ e^{\dagger}(-q)e^{-iq\delta_{be}} + f(q)e^{-iq\delta_{be}} \right] | 0 \rangle \\
- \frac{J_{12}}{N_{uc}} \sum_q \left< 0 \right| a(q) \left( \frac{1}{\xi} c(-q) \right) \left| 0 \right> \langle 0 | f^\dagger(q)e^{iq\delta_{de}} \left[ e^{\dagger}(-q)e^{-iq\delta_{de}} + f(q)e^{-iq\delta_{de}} \right] | 0 \rangle \\
- \frac{J_{12}}{N_{uc}} \sum_q \left< 0 \right| a(q) \left( \frac{1}{\xi} d^\dagger(q) \right) \left| 0 \right> \langle 0 | f^\dagger(q)e^{iq\delta_{de}} \left[ e^{\dagger}(-q)e^{-iq\delta_{de}} + f(q)e^{-iq\delta_{de}} \right] | 0 \rangle .
\]

For the CuII subsystem we have the usual relations

\[
e(q) = l_q \eta(q) - m_q \delta^{\dagger}(-q) , \\
f^{\dagger}(-q) = -m_q \eta(q) + l_q \delta^{\dagger}(-q) ,
\]

(B10)
where $\eta(q)$ and $\delta(q)$ are the normal mode operators for the CuII subsystem and $l_q$ and $m_q$ are given in Eq. (E11). In the ground state we evaluate the averages to get

$$A_i = \frac{J_{12} S}{N_{uc}} \sum_q \left< 0 \left| \alpha^\dagger(q) \frac{1}{\epsilon} \alpha(q) \right| 0 \right> \left[ -l_q m_q e^{-iq \delta_{ae}} + l^2_q e^{-iq \delta_{af}} \right] e^{iq \delta_{af}}$$

$$- \frac{J_{12} S}{N_{uc}} \sum_q \left< 0 \left| b(-q) \frac{1}{\epsilon} a(q) \right| 0 \right> \left[ -l_q m_q e^{-iq \delta_{be}} + l^2_q e^{-iq \delta_{bf}} \right] e^{iq \delta_{bf}}$$

$$- \frac{J_{12} S}{N_{uc}} \sum_q \left< 0 \left| c(-q) \frac{1}{\epsilon} a(q) \right| 0 \right> \left[ -l_q m_q e^{-iq \delta_{ce}} + l^2_q e^{-iq \delta_{cf}} \right] e^{iq \delta_{cf}}$$

$$- \frac{J_{12} S}{N_{uc}} \sum_q \left< 0 \left| d^\dagger(q) \frac{1}{\epsilon} a(q) \right| 0 \right> \left[ -l_q m_q e^{-iq \delta_{de}} + l^2_q e^{-iq \delta_{df}} \right] e^{iq \delta_{df}}$$

$$- \frac{J_{12} S}{N_{uc}} \sum_q \left< 0 \left| a(q) \frac{1}{\epsilon} b(-q) \right| 0 \right> \left[ -l_q m_q e^{-iq \delta_{be}} + l^2_q e^{-iq \delta_{bf}} \right] e^{iq \delta_{bf}}$$

$$- \frac{J_{12} S}{N_{uc}} \sum_q \left< 0 \left| a(q) \frac{1}{\epsilon} c(-q) \right| 0 \right> \left[ -l_q m_q e^{-iq \delta_{ce}} + l^2_q e^{-iq \delta_{cf}} \right] e^{iq \delta_{cf}}$$

$$- \frac{J_{12} S}{N_{uc}} \sum_q \left< 0 \left| a(q) \frac{1}{\epsilon} d^\dagger(q) \right| 0 \right> \left[ -l_q m_q e^{-iq \delta_{de}} + l^2_q e^{-iq \delta_{df}} \right] e^{iq \delta_{df}}$$

$$= \frac{J_{12} S}{N_{uc}} \sum_q \left< 0 \left| a^\dagger(q) \frac{1}{\epsilon} a(q) \right| 0 \right> \left[ -l_q m_q e^{iqa} + l^2_q \right]$$

$$- \frac{J_{12} S}{N_{uc}} \sum_q \left< 0 \left| b(-q) \frac{1}{\epsilon} a(q) \right| 0 \right> \left[ -l_q m_q e^{-iqa/2} + l^2_q e^{iqa/2} \right] e^{iqa/2}$$

$$- \frac{J_{12} S}{N_{uc}} \sum_q \left< 0 \left| c(-q) \frac{1}{\epsilon} a(q) \right| 0 \right> \left[ -l_q m_q e^{iqa/2} + l^2_q e^{-iqa/2} \right] e^{iqa/2}$$

$$- \frac{J_{12} S}{N_{uc}} \sum_q \left< 0 \left| d^\dagger(q) \frac{1}{\epsilon} a(q) \right| 0 \right> \left[ -l_q m_q + l^2_q e^{iqa} \right]$$

$$- \frac{J_{12} S}{N_{uc}} \sum_q \left< 0 \left| a(q) \frac{1}{\epsilon} b(-q) \right| 0 \right> \left[ -l_q m_q e^{-iqa/2} + l^2_q e^{iqa/2} \right] e^{iqa/2}$$

$$- \frac{J_{12} S}{N_{uc}} \sum_q \left< 0 \left| a(q) \frac{1}{\epsilon} c(-q) \right| 0 \right> \left[ -l_q m_q e^{iqa/2} + l^2_q e^{-iqa/2} \right] e^{iqa/2}$$

$$- \frac{J_{12} S}{N_{uc}} \sum_q \left< 0 \left| a(q) \frac{1}{\epsilon} d^\dagger(q) \right| 0 \right> \left[ s - l_q m_q + l^2_q e^{iqa} \right] .$$

For the CuI subsystem we have normal modes via the transformations,

$$a(q) = (1/\sqrt{2})[a_+(q) + a_-(q)]$$
\[ d(\mathbf{q}) = (1/\sqrt{2})[a_+(\mathbf{q}) - a_-(\mathbf{q})] \]
\[ b(\mathbf{q}) = (1/\sqrt{2})[b_+(\mathbf{q}) + b_-(\mathbf{q})] \]
\[ c(\mathbf{q}) = (1/\sqrt{2})[b_+(\mathbf{q}) - b_-(\mathbf{q})] . \]  

(B12)

In terms of these operators (in the order \(a_+, b_+, a_-, b_-\)) we have the matrices \( \mathbf{A} \) and \( \mathbf{B} \):

\[
\frac{\mathbf{A}(\mathbf{q})}{S} = \begin{bmatrix}
4J + 2J_3 & 0 & 0 & 0 \\
0 & 4J + 2J_3 & 0 & 0 \\
0 & 0 & 4J + 2J_3 & 0 \\
0 & 0 & 0 & 4J + 2J_3
\end{bmatrix} 
\]  

(B13)

and \( \mathbf{B}(\mathbf{q})/S \) as

\[
\begin{bmatrix}
0 & 2J(c_+ + c_-) + 2J_3c_z & 0 & 0 \\
2J(c_+ + c_-) + 2J_3c_z & 0 & 0 & 0 \\
0 & 0 & 0 & 2J(c_+ - c_-) + 2J_3c_z \\
0 & 0 & 2J(c_+ - c_-) + 2J_3c_z & 0
\end{bmatrix}, \quad (B14)
\]

where

\[
c_+ = \cos[a(q_x + q_y)/2], \quad c_- = \cos[a(q_x - q_y)/2], \quad c_z = \cos(q_zc/2) . \]  

(B15)

Now each sector has relations analogous to the CuII's:

\[
a_\sigma(\mathbf{q}) = l_{\sigma,q}a_\sigma(\mathbf{q}) - m_{\sigma,q}\beta^\dagger_\sigma(-\mathbf{q}) \\
b^\dagger_\sigma(-\mathbf{q}) = -m_{\sigma,q}a_\sigma(\mathbf{q}) + l_{\sigma,q}\beta_\sigma(-\mathbf{q}) , \quad (B16)
\]

where \(\alpha_\sigma(\mathbf{q})\) and \(\beta_\sigma(\mathbf{q})\) are the normal mode operators, and

\[
l^2_{\sigma,q} = \frac{A + E_\sigma(\mathbf{q})}{2E_\sigma(\mathbf{q})} , \quad m^2_{\sigma,q} = \frac{A - E_\sigma(\mathbf{q})}{2E_\sigma(\mathbf{q})} , \quad l_{\sigma,q}m_{\sigma,q} = \frac{B_\sigma(\mathbf{q})}{2E_\sigma(\mathbf{q})} . \]  

(B17)

Here

\[
E_\sigma(\mathbf{q})^2 = A^2 - B_\sigma(\mathbf{q})^2 , \quad (B18)
\]

where
\[ A = 4J + 2J_3 \]

\[ B_\sigma(q) = 2J \left( \cos[(q_x + q_y)a/2] + \sigma \cos[(q_x - q_y)a/2] \right) + 2J_3 \cos(q_z c) , \]

so that

\[ B_+(q) = 2J_3 \cos(q_z c) + 4J \cos(q_x a/2) \cos(q_y a/2) \]

\[ B_-(q) = 2J_3 \cos(q_z c) - 4J \sin(q_x a/2) \sin(q_y a/2) . \]

Thus

\[ \langle 0|a^\dagger(q) \frac{1}{\xi} a(q)|0 \rangle = \frac{1}{2S} \sum_\sigma \langle 0|a^\dagger_\sigma(q) a_\sigma(q)|0 \rangle E_\sigma(q)^{-1} \]

\[ = \frac{1}{2S} \sum_\sigma m_{\sigma q}^2 E_\sigma(q)^{-1} = \sum_\sigma \frac{A - E_\sigma(q)}{4SE_\sigma(q)^2} . \]

Similarly

\[ \langle 0|a(q) \frac{1}{\xi} a^\dagger(q)|0 \rangle = \frac{1}{2S} \sum_\sigma \langle 0|a_\sigma(q) a^\dagger_\sigma(q)|0 \rangle E_\sigma(q)^{-1} \]

\[ = \frac{1}{2S} \sum_\sigma \sum_{\sigma q} m_{\sigma q}^2 E_\sigma(q)^{-1} = \sum_\sigma \frac{A + E_\sigma(q)}{4SE_\sigma(q)^2} \]

\[ \langle 0|a(q) \frac{1}{\xi} d^\dagger(q)|0 \rangle = \frac{1}{2S} \sum_\sigma \langle 0|a_\sigma(q) d^\dagger_\sigma(q)|0 \rangle E_\sigma(q)^{-1} \]

\[ = \frac{1}{2S} \sum_\sigma \sum_{\sigma q} m_{\sigma q}^2 E_\sigma(q)^{-1} = \sum_\sigma \frac{A + E_\sigma(q)}{4SE_\sigma(q)^2} \]

\[ \langle 0|b(-q) \frac{1}{\xi} a(q)|0 \rangle = \frac{1}{2S} \sum_\sigma \langle 0|b_\sigma(-q) a_\sigma(q)|0 \rangle E_\sigma(q)^{-1} \]

\[ = -\frac{1}{2S} \sum_\sigma l_{\sigma q} m_{\sigma q} E_\sigma(q)^{-1} = -\sum_\sigma \frac{B_\sigma(q)}{4SE_\sigma(q)^2} \]

\[ \langle 0|a(q) \frac{1}{\xi} b(-q)|0 \rangle = \frac{1}{2S} \sum_\sigma \langle 0|a_\sigma(q) b_\sigma(-q)|0 \rangle E_\sigma(q)^{-1} \]

\[ = -\frac{1}{2S} \sum_\sigma l_{\sigma q} m_{\sigma q} E_\sigma(q)^{-1} = -\sum_\sigma \frac{B_\sigma(q)}{4SE_\sigma(q)^2} \]
\[
\langle 0 | c(-q) \frac{1}{E} a(q) | 0 \rangle = \frac{1}{2S} \sum_{\sigma} \sigma \langle 0 | b_{\sigma}(-q) a_{\sigma}(q) | 0 \rangle E_{\sigma}(q)^{-1}
\]
\[
= -\frac{1}{2S} \sum_{\sigma} \sigma l_{\sigma, q} m_{\sigma q} E_{\sigma}(q)^{-1} = - \sum_{\sigma} \frac{B_{\sigma}(q)}{4SE_{\sigma}(q)^2}
\]  
(B27)
\[
\langle 0 | a(q) \frac{1}{E} c(-q) | 0 \rangle = \frac{1}{2S} \sum_{\sigma} \sigma \langle 0 | a_{\sigma}(q) b_{\sigma}(-q) | 0 \rangle E_{\sigma}(q)^{-1}
\]
\[
= -\frac{1}{2S} \sum_{\sigma} \sigma l_{\sigma, q} m_{\sigma q} E_{\sigma}(q)^{-1} = - \sum_{\sigma} \frac{B_{\sigma}(q)}{4SE_{\sigma}(q)^2}.
\]  
(B28)

Then
\[
A_1 = -\frac{J_{12}}{8N_{uc}} \sum_{q} \sum_{\sigma} [E_{\sigma}(q)^2 \epsilon(q)]^{-1}
\]
\[
[A - E_{\sigma}(q)]\{-\gamma(q)e^{i q x a} + 1 + \epsilon(q)\} + [A + E_{\sigma}(q)]\{-\gamma(q)e^{i q x a} + 1 - \epsilon(q)\}
\]
\[
- B_{\sigma}(q)[-\gamma(q)e^{-i q x a} + (1 + \epsilon(q))e^{i q x a/2}] e^{i q x a/2}
\]
\[
- B_{\sigma}(q)[-\gamma(q)e^{-i q x a} + (1 - \epsilon(q))e^{i q x a/2}] e^{-i q x a/2}
\]
\[
- \sigma B_{\sigma}(q)[-\gamma(q)e^{i q y a/2} + (1 + \epsilon(q))e^{-i q y a/2}] e^{-i q y a/2}
\]
\[
+ \sigma[A - E_{\sigma}(q)][-\gamma(q) + (1 + \epsilon(q))e^{i q y a}]
\]
\[
+ \sigma[A + E_{\sigma}(q)][-\gamma(q) + (1 - \epsilon(q))e^{i q y a}]
\]  
(B29)

where
\[
\gamma(q) = \frac{1}{2} \cos(q_x a) + \cos(q_y a)
\]  
(B30)

and
\[
\epsilon(q)^2 = 1 - \gamma(q)^2.
\]  
(B31)

We use the fact that $J_3 \ll J$. Only if a sum is divergent will it make a difference if we retain nonzero $J_3$. So we tentatively assume no divergences and write

\[
B_+(q) = 4J \cos(q_x a/2) \cos(q_y a/2)
\]  
(B32a)
\[
B_-(q) = -4J \sin(q_x a/2) \sin(q_y a/2).
\]  
(B32b)
We now simplify Eq. (B29). We note that under the sum over wavevectors we can replace \( \exp(iq_xa) \) by \( \gamma(q) \). Let us apply the same reasoning to \( \exp[i(q_x \pm q_y)a/2] \):

\[
\exp[i(q_x \pm q_y)a/2] = \cos(q_xa/2) \cos(q_ya/2) \mp \sin(q_xa/2) \sin(q_ya/2) \\
+ i \left( \sin(q_xa/2) \cos(q_ya/2) a \pm \cos(q_xa/2) \sin(q_ya/2) \right). \tag{B33}
\]

After summation over wavevectors the imaginary parts will drop out. So

\[
\exp[i(q_x + q_y)a/2] = \cos(q_xa/2) \cos(q_ya/2) - \sin(q_xa/2) \sin(q_ya/2)
\]

\[
= \left( \frac{1}{4J} \right) \sum_\sigma B_\sigma(q). \tag{B34a}
\]

\[
\exp[i(q_x - q_y)a/2] = \cos(q_xa/2) \cos(q_ya/2) + \sin(q_xa/2) \sin(q_ya/2)
\]

\[
= \left( \frac{1}{4J} \right) \sum_\sigma \sigma B_\sigma(q). \tag{B34b}
\]

In this connection note that sums which are proportional to \( B_+(q)B_-(q) \) vanish. So

\[
A_1 = -\frac{J_1}{8N_{uc}} \sum_q \sum_\sigma \left( \frac{1}{E_\sigma(q)^2\epsilon(q)} \right) \left( [A - E_\sigma(q)] [-\gamma(q)^2 + 1 + \epsilon(q)] + [A + E_\sigma(q)] [-\gamma(q)^2 + 1 - \epsilon(q)] \\
+ \sigma [A - E_\sigma(q)] [\gamma(q) \epsilon(q)] + \sigma [A + E_\sigma(q)] [-\gamma(q) \epsilon(q)] \\
- \left( \frac{B_\sigma(q)}{4J} \right) [-\gamma(q) \sigma B_\sigma(q) + (1 + \epsilon(q)) B_\sigma(q)] \\
- \left( \frac{B_\sigma(q)}{4J} \right) [-\gamma(q) \sigma B_\sigma(q) + (1 - \epsilon(q)) B_\sigma(q)] \\
- \left( \frac{\sigma B_\sigma(q)}{4J} \right) [-\gamma(q) B_\sigma(q) + (1 + \epsilon(q)) \sigma B_\sigma(q)] \\
- \left( \frac{\sigma B_\sigma(q)}{4J} \right) [-\gamma(q) B_\sigma(q) + (1 - \epsilon(q)) \sigma B_\sigma(q)] \right) \\
= -\frac{J_1}{8N_{uc}} \sum_q \sum_\sigma \left( \frac{1}{E_\sigma(q)^2\epsilon(q)} \right) \left( 2A\epsilon(q)^2 - 2E_\sigma(q)\epsilon(q) - 2E_\sigma(q)\epsilon(q)\sigma \gamma(q) - J^{-1}B_\sigma(q)^2 [1 - \sigma \gamma(q)] \right). \tag{B35}
\]

Now we must understand how the wavevector sums are to be done. The unit cell is

\[
a_1 = a\hat{x} + a\hat{y}, \quad a_2 = -a\hat{x} + a\hat{y}, \tag{B36}
\]

Thus the reciprocal lattice vectors are
\[ G_1 = (\pi/a)(\hat{x} + \hat{y}) \, , \quad G_2 = (\pi/a)(-\hat{x} + \hat{y}) \, . \]  

Thus the sums are carried over the first zone, shown below in Fig. 7.

2. \( A_2 \)

Thus

\[ A_2 = - \left\langle 0 \left| V_{\text{II}-\text{I}} \frac{1}{\mathcal{E}} a_m e_n \right| 0 \right\rangle - \left\langle 0 \left| a_m e_n \frac{1}{\mathcal{E}} V_{\text{II}-\text{I}} \right| 0 \right\rangle , \]  

where we invoke perturbation theory relative to decoupled CuI and CuII subsystems. As for \( A_1 \) effectively we have Eq. (B4). Thus

\[ A_2 = -J_{12S} \sum_{i \in a} \left\langle 0 \left| a_i \frac{1}{\mathcal{E}} a_m \right| 0 \right\rangle \left\langle 0 \left| [e_{i+\delta_{ce}}^\dagger + f_{i+\delta_{fe}}] e_{m+\delta_{ce}} \right| 0 \right\rangle \]

\[ -J_{12S} \sum_{i \in b} \left\langle 0 \left| b_i \frac{1}{\mathcal{E}} a_m \right| 0 \right\rangle \left\langle 0 \left| [e_{i+\delta_{be}}^\dagger + f_{i+\delta_{ef}}] e_{m+\delta_{be}} \right| 0 \right\rangle \]

\[ -J_{12S} \sum_{i \in c} \left\langle 0 \left| c_i \frac{1}{\mathcal{E}} a_m \right| 0 \right\rangle \left\langle 0 \left| [e_{i+\delta_{ce}}^\dagger + f_{i+\delta_{fe}}] e_{m+\delta_{ce}} \right| 0 \right\rangle \]

\[ -J_{12S} \sum_{i \in d} \left\langle 0 \left| d_i \frac{1}{\mathcal{E}} a_m \right| 0 \right\rangle \left\langle 0 \left| [e_{i+\delta_{be}}^\dagger + f_{i+\delta_{ef}}] e_{m+\delta_{be}} \right| 0 \right\rangle \]

\[ -J_{12S} \sum_{i \in a} \left\langle 0 \left| a_m \frac{1}{\mathcal{E}} a_i^\dagger \right| 0 \right\rangle \left\langle 0 \left| e_{m+\delta_{be}} [e_{i+\delta_{be}}^\dagger + f_{i+\delta_{ef}}] \right| 0 \right\rangle \]

\[ -J_{12S} \sum_{i \in b} \left\langle 0 \left| a_m \frac{1}{\mathcal{E}} b_i \right| 0 \right\rangle \left\langle 0 \left| e_{m+\delta_{be}} [e_{i+\delta_{be}}^\dagger + f_{i+\delta_{ef}}] \right| 0 \right\rangle \]

\[ -J_{12S} \sum_{i \in c} \left\langle 0 \left| a_m \frac{1}{\mathcal{E}} c_i \right| 0 \right\rangle \left\langle 0 \left| e_{m+\delta_{be}} [e_{i+\delta_{be}}^\dagger + f_{i+\delta_{ef}}] \right| 0 \right\rangle \]

\[ -J_{12S} \sum_{i \in d} \left\langle 0 \left| a_m \frac{1}{\mathcal{E}} d_i \right| 0 \right\rangle \left\langle 0 \left| e_{m+\delta_{be}} [e_{i+\delta_{be}}^\dagger + f_{i+\delta_{ef}}] \right| 0 \right\rangle . \]  

Then

\[ A_2 = - \frac{J_{12S}}{N_{uc}^2} \sum_{q, k} \sum_{i \in a} \left\langle 0 \left| a_i^\dagger (q) \frac{1}{\mathcal{E}} a(q) \right| 0 \right\rangle e^{i(q+k) \cdot \mathbf{r}_{im}} \]

\[ \times \left\langle 0 \left| e^i(k) e^{ik \delta_{be}} + f(-k) e^{ik \delta_{fe}} \right| e(k) e^{-ik \delta_{be}} \right| 0 \right\rangle \]

\[ - \frac{J_{12S}}{N_{uc}^2} \sum_{q, k} \sum_{i \in b} \left\langle 0 \left| b(-q) \frac{1}{\mathcal{E}} a(q) \right| 0 \right\rangle e^{i(q+k) \cdot \mathbf{r}_{im}} \]

\[ \times \left\langle 0 \left| e^i(k) e^{ik \delta_{be}} + f(-k) e^{ik \delta_{fe}} \right| e(k) e^{-ik \delta_{be}} \right| 0 \right\rangle \]
\begin{align}
- \frac{J_{12} S}{N_{uc}} \sum_{q,k} \sum_{i \in c} \langle 0 | e(-q) \frac{1}{\mathcal{E}} a(q) | 0 \rangle e^{i(q+k) r_{im}} \\
\times \langle 0 | [e^{\dagger}(k)e^{ik \delta_{ae}} + f(-k)e^{ik \delta_{cf}}] e(k)e^{-ik \delta_{a'e}} | 0 \rangle \\
- \frac{J_{12} S}{N_{uc}} \sum_{q,k} \sum_{i \in d} \langle 0 | d^{\dagger}(q) \frac{1}{\mathcal{E}} a(q) | 0 \rangle e^{i(q+k) r_{im}} \\
\times \langle 0 | [e^{\dagger}(k)e^{ik \delta_{ae}} + f(-k)e^{ik \delta_{de}}] e(k)e^{-ik \delta_{ae'}} | 0 \rangle \\
- \frac{J_{12} S}{N_{uc}} \sum_{q,k} \sum_{i \in a} \langle 0 | a(q) \frac{1}{\mathcal{E}} d^{\dagger}(q) | 0 \rangle e^{i(q+k) r_{im}} \\
\times \langle 0 | [e^{\dagger}(k)e^{ik \delta_{ae}} + f(-k)e^{ik \delta_{de}}] e(k)e^{-ik \delta_{ae'}} | 0 \rangle \\
\times \langle 0 | e(k)e^{-ik \delta_{ae}} [e^{\dagger}(k)e^{ik \delta_{de}} + f(-k)e^{ik \delta_{ae'}}] | 0 \rangle.
\end{align}

Doing the sum over $i$ we get

\begin{align}
A_2 &= - \frac{J_{12} S}{N_{uc}} \sum_{q} \langle 0 | a^{\dagger}(q) \frac{1}{\mathcal{E}} a(q) | 0 \rangle \langle 0 | [e^{\dagger}(-q)e^{-iq \delta_{ae}} + f(q)e^{-iq \delta_{ae'}}] e(-q)e^{iq \delta_{ae'}} | 0 \rangle \\
- \frac{J_{12} S}{N_{uc}} \sum_{q} \langle 0 | b(-q) \frac{1}{\mathcal{E}} a(q) | 0 \rangle \langle 0 | [e^{\dagger}(-q)e^{-iq \delta_{de}} + f(q)e^{-iq \delta_{de'}}] e(-q)e^{iq \delta_{de'}} | 0 \rangle \\
- \frac{J_{12} S}{N_{uc}} \sum_{q} \langle 0 | c(-q) \frac{1}{\mathcal{E}} a(q) | 0 \rangle \langle 0 | [e^{\dagger}(-q)e^{-iq \delta_{de}} + f(q)e^{-iq \delta_{de'}}] e(-q)e^{iq \delta_{de'}} | 0 \rangle \\
- \frac{J_{12} S}{N_{uc}} \sum_{q} \langle 0 | d^{\dagger}(-q) \frac{1}{\mathcal{E}} a(q) | 0 \rangle \langle 0 | [e^{\dagger}(-q)e^{-iq \delta_{de}} + f(q)e^{-iq \delta_{de'}}] e(-q)e^{iq \delta_{de'}} | 0 \rangle \\
- \frac{J_{12} S}{N_{uc}} \sum_{q} \langle 0 | a(q) \frac{1}{\mathcal{E}} a^{\dagger}(q) | 0 \rangle \langle 0 | e(-q)e^{iq \delta_{ae'}} [e^{\dagger}(-q)e^{-iq \delta_{ae}} + f(q)e^{-iq \delta_{ae'}}] | 0 \rangle \\
- \frac{J_{12} S}{N_{uc}} \sum_{q} \langle 0 | a(q) \frac{1}{\mathcal{E}} b(-q) | 0 \rangle \langle 0 | e(-q)e^{iq \delta_{ae'}} [e^{\dagger}(-q)e^{-iq \delta_{de}} + f(q)e^{-iq \delta_{de'}}] | 0 \rangle \\
- \frac{J_{12} S}{N_{uc}} \sum_{q} \langle 0 | a(q) \frac{1}{\mathcal{E}} c(-q) | 0 \rangle \langle 0 | e(-q)e^{iq \delta_{ae'}} [e^{\dagger}(-q)e^{-iq \delta_{de}} + f(q)e^{-iq \delta_{de'}}] | 0 \rangle \\
- \frac{J_{12} S}{N_{uc}} \sum_{q} \langle 0 | a(q) \frac{1}{\mathcal{E}} d^{\dagger}(q) | 0 \rangle \langle 0 | e(-q)e^{iq \delta_{ae'}} [e^{\dagger}(-q)e^{-iq \delta_{de}} + f(q)e^{-iq \delta_{de'}}] | 0 \rangle.
\end{align}

This is 54
\[ A_2 = -\frac{J_{12}}{4N_{uc}} \sum_{\sigma} \left( E_\sigma(q)^{-2} \right) \left[ A - E_\sigma(q) \right] \left[ m_q^2 - l_q m_q e^{-i q x q} \right] + \left[ A + E_\sigma(q) \right] \left[ l_q^2 - l_q m_q e^{i q x q} \right] \\
- B\sigma(q) \left[ \frac{e^{-i(q_x+q_y)a/2} m_q^2 - l_q m_q e^{i(q_x+q_y)a/2}}{q^2} - B\sigma(q) \left[ \frac{e^{-i(q_x+q_y)a/2} l_q^2 - l_q m_q e^{i(q_x+q_y)a/2}}{q^2} \right] \right] \\
- \sigma B\sigma(q) \left[ e^{i(q_x+q_y)a/2} m_q^2 - e^{-i(q_x+q_y)a/2} l_q m_q \right] - \sigma B\sigma(q) \left[ e^{i(q_x+q_y)a/2} l_q^2 - e^{-i(q_x+q_y)a/2} l_q m_q \right] \\
+ \sigma [ A - E_\sigma ] \left[ e^{-i q x q} m_q^2 - l_q m_q \right] + \sigma [ A + E_\sigma ] \left[ e^{-i q x q} m_q^2 - l_q m_q \right] \right] . \] (B43)

Making the same replacements as in \( A_1 \) we get

\[ A_2 = -\frac{J_{12}}{4N_{uc}} \sum_{\sigma} \sum_{q} E_\sigma(q)^{-2} \left( \left[ A - E_\sigma(q) \right] \left[ m_q^2 - l_q m_q \gamma(q) \right] + \left[ A + E_\sigma(q) \right] \left[ l_q^2 - l_q m_q \gamma(q) \right] \right) \\
- B\sigma(q) B\sigma(q)(4J)^{-1} \left[ m_q^2 - \sigma l_q m_q \right] - B\sigma(q) B\sigma(q)(4J)^{-1} \left[ l_q^2 - \sigma l_q m_q \right] \\
- \sigma B\sigma(q) B\sigma(q)(4J)^{-1} \left[ m_q^2 - l_q m_q \right] - \sigma B\sigma(q) B\sigma(q)(4J)^{-1} \left[ l_q^2 - l_q m_q \right] \\
+ \sigma [ A - E_\sigma ] \left[ \gamma(q) m_q^2 - l_q m_q \right] + \sigma [ A + E_\sigma ] \left[ \gamma(q) l_q^2 - l_q m_q \right] \right] . \] (B44)

This is
\[ A_2 = -\frac{J_{12}}{8N_{uc}} \sum_{q} \sum_{\sigma} E_\sigma(q)^{-2} \epsilon(q)^{-1} \left( \right. \]
\[
\left. [A - E_\sigma(q)][1 - \epsilon(q) - \gamma(q)^2] + [A + E_\sigma(q)][1 + \epsilon(q) - \gamma(q)^2] \right. \]
\[
- \left( \frac{B_\sigma(q)^2}{4J} \right) [2 - 2\sigma\gamma(q)] - \left( \frac{\sigma B_\sigma(q)^2}{4J} \right) [2\sigma - 2\gamma(q)] \right. \]
\[
+ \sigma [A - E_\sigma(q)] \gamma(q)[1 - \epsilon(q)] - \gamma(q)] + \sigma [A + E_\sigma(q)] \left( \gamma(q)[1 + \epsilon(q)] + \gamma(q) \right) \right) . \tag{B45} \]

So

\[ A_2 = -\frac{J_{12}}{8N_{uc}} \sum_{q} \sum_{\sigma} \left( \frac{1}{E_\sigma(q)^2} \epsilon(q) \right) \left( 2A\epsilon(q)^2 + 2E_\sigma(q)\epsilon(q) + 2E_\sigma(q)\sigma\gamma(q)\epsilon(q) \right. \]
\[
\left. - B_\sigma(q)^2 J^{-1} [1 - \sigma\gamma(q)] \right) \right) . \tag{B46} \]

3. Summary

So

\[ A_1 = - \left( \frac{J_{12}}{2J} \right) (C_\alpha - C_\beta) \] \hfill (B47a)

\[ A_2 = - \left( \frac{J_{12}}{2J} \right) (C_\alpha + C_\beta) \] \hfill (B47b)

where

\[ C_\alpha = \frac{J}{4N_{uc}} \sum_{q} \sum_{\sigma} \left( \frac{1}{E_\sigma(q)^2} \right) \left( 2A\epsilon(q)^2 - B_\sigma(q)^2 J^{-1} [1 - \sigma\gamma(q)] \right) \] \hfill (B48a)

\[ C_\beta = \frac{J}{4N_{uc}} \sum_{q} \sum_{\sigma} \left( \frac{1}{E_\sigma(q)} \right) \left( 2[1 + \sigma\gamma(q)] \right) \] \hfill (B48b)

If we extend the sum over \(-\pi/a < q_x, q_y < \pi/a\), then we may write these as

\[ C_\alpha = \frac{J}{4N_{uc}} \sum_{q} \left( \frac{1}{E_+(q)^2} \right) \left( 2A\epsilon(q)^2 - B_+(q)^2 J^{-1} [1 - \gamma(q)] \right) \] \hfill (B49a)

\[ C_\beta = \frac{J}{4N_{uc}} \sum_{q} \left( \frac{1}{E_+(q)} \right) \left( 2[1 + \gamma(q)] \right) \] \hfill (B49b)

Of course, note that now \(\sum_q = 2N_{uc}\). So it is convenient to introduce the notation \(\langle \rangle_q\) to denote \((2N_{uc})^{-1} \sum_q\). Then

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\[ C_\alpha = \frac{J}{2} \left\langle \frac{1}{E_+(\mathbf{q})^2 \epsilon_\mathbf{q}} \left( 2 A_\epsilon(\mathbf{q})^2 - B_+(\mathbf{q})^2 J^{-1} [1 - \gamma(\mathbf{q})] \right) \right\rangle_\mathbf{q} \]  
\[ C_\beta = \frac{J}{2} \left\langle \frac{1}{E_+(\mathbf{q})} \left( 2 [1 + \gamma(\mathbf{q})] \right) \right\rangle_\mathbf{q} \]  

Or

\[ C_\alpha = \frac{1}{4} \left\langle \frac{1 - \gamma(\mathbf{q})^2 - 2 \cos^2(\frac{a q_x}{2}) \cos^2(\frac{a q_y}{2}) [1 - \gamma(\mathbf{q})]}{[1 - \cos^2(\frac{a q_x}{2}) \cos^2(\frac{a q_y}{2})] \sqrt{1 - \gamma(\mathbf{q})^2}} \right\rangle_\mathbf{q} \]  
\[ C_\beta = \frac{1}{4} \left\langle \frac{1 + \gamma(\mathbf{q})}{\sqrt{1 - \cos^2(\frac{a q_x}{2}) \cos^2(\frac{a q_y}{2})}} \right\rangle_\mathbf{q} . \]

In the approximation that \( \gamma(\mathbf{q}) = 0 \), etc. \( C_\alpha = C_\beta = \frac{1}{4} \). Numerical evaluation yields

\[ C_\alpha = 0.1686 , \quad C_\beta = 0.4210 . \]  

**APPENDIX C: IN-PLANE CUI – CUI INTERACTION**

Here we reproduce by perturbation theory the gap found phenomenologically by Yildirim et al.\(^{13}\) We treat an antiferromagnet on a square lattice (of lattice constant \( a \)), in which there are two sublattices, \( a \) and \( b \). The lattice is shown in the Fig. 6 with the magnetic unit cell within dashed lines. The magnetic unit cell has basis vectors

\[ \mathbf{a}_1 = a \hat{x} + a \hat{y} , \]
\[ \mathbf{a}_2 = -a \hat{x} + a \hat{y} . \]  

We transform to bosons using Eq. (13).

First we consider terms \( \mathcal{H} \) in the Hamiltonian which are quadratic in boson operators. We write

\[ \mathcal{H} = \mathcal{H}_J + \mathcal{H}_\delta . \]  

Here

\[ \mathcal{H}_J = 4 J S \sum_\mathbf{q} \left( a^\dagger(\mathbf{q})a(\mathbf{q}) + b^\dagger(\mathbf{q})b(\mathbf{q}) + \gamma(\mathbf{q})[a^\dagger(\mathbf{q})b^\dagger(-\mathbf{q}) + a(\mathbf{q})b(-\mathbf{q})] \right) , \]  

\[ 57 \]
with
\[ \gamma(q) = \frac{1}{2} \left[ \cos q_x a + \cos q_y a \right]. \tag{C4} \]
and the sum over wave vectors is over the Brillouin zone associated with the magnetic unit cell. Also
\[ \mathcal{H}_\delta = \delta J_1 S \sum_k \left( c_x(k) - c_y(k) \right) \left[ a(k) + a^\dagger(-k) \right] \left[ b^\dagger(k) + b(-k) \right], \tag{C5} \]
where \( c_x(k) = \cos k_x a \) and \( c_y(k) = \cos k_y a \).

Since the effect we wish to treat involves energies of relative order \((1/S)\), we now consider the fourth-order terms, \( V_4 \), in the boson Hamiltonian, which we write as
\[ V_4 = V_J + V_\delta, \tag{C6} \]
where
\[ V_J = -\frac{2}{J} \sum_{\langle ij \rangle} b^\dagger_j (a^\dagger_i + b_j)^2 a_i, \tag{C7} \]
where \( \langle ij \rangle \) indicates that \( i \) is summed over a sites and \( j \) over nearest neighboring b sites and
\[ V_\delta = \delta J_1 \sum_{\langle ij \rangle} \sigma_\delta \left[ -\frac{1}{2} a^\dagger_i a^\dagger_i a_i (b_j^\dagger + b_j) - \frac{1}{2} (a_i^\dagger + a_i) b_j^\dagger b_j + a_i^\dagger a_i b_j^\dagger b_j \right], \tag{C8} \]
where \( \sigma_\delta \) is +1 for \( x \) bonds and -1 for \( y \) bonds.

We construct the effective quadratic Hamiltonian by taking all possible averages of pairs of operators out of the fourth order terms. Thus we have the effective quadratic terms
\[ \Delta H_J = -\frac{1}{2} J \sum_{\langle ij \rangle} \left[ a_i^\dagger b_j^\dagger (a_j + a_i^\dagger)^2 + 2 a_i (b_j + a_i^\dagger) b_j^\dagger (b_j + a_i^\dagger) \right] \]
\[ +2 b_j^\dagger (b_j + a_i^\dagger) (b_j + a_i^\dagger) a_i + (b_j + a_i^\dagger)^2 \langle a_i b_j^\dagger \rangle \] \tag{C9} 
and
\[ \Delta H_\delta = \delta J_1 \sum_{\langle ij \rangle} \sigma_\delta \left[ -\frac{1}{2} a_i^\dagger (b_j^\dagger + b_j) \langle a_j^\dagger a_i^\dagger \rangle - \frac{1}{2} a_i^\dagger a_i \langle b_j^\dagger (b_j + b_j) \rangle - \frac{1}{4} a_i^\dagger a_i \langle a_i (b_j^\dagger + b_j) \rangle \right. \]
\[ -\frac{1}{2} a_i^\dagger + a_i \langle b_j^\dagger b_j \rangle - \frac{1}{2} a_i^\dagger + a_i \langle b_j b_j \rangle - \frac{1}{4} b_j^\dagger b_j \langle a_i^\dagger + a_i \rangle \]
\[ -\frac{1}{4} (a_i^\dagger + a_i) b_j^\dagger \langle b_j b_j \rangle - \frac{1}{4} b_j b_j \langle (a_i^\dagger + a_i) b_j^\dagger \rangle + a_i^\dagger a_i \langle b_j b_j \rangle + b_j^\dagger b_j \langle a_i^\dagger a_i \rangle \]
\[ + a_i^\dagger b_j \langle b_j a_i^\dagger \rangle + a_i^\dagger b_j^\dagger \langle a_i b_j \rangle + b_j a_i \langle b_j a_i^\dagger \rangle \right]. \tag{C10} \]
where $\langle X \rangle$ denotes an average with respect to the quadratic Hamiltonian.

Since the quadratic Hamiltonian is real and Hermitian we can equate averages like $\langle a_i^\dagger b_j^\dagger \rangle$ and $\langle a_i b_j \rangle$. Also at this order of $(1/\mathcal{S})$ we only need keep Hermitian contributions to the effective Hamiltonian. Therefore we write

$$
\Delta H_J = -\frac{1}{4} J \sum_{(ij)} \left[ (a_i b_j^\dagger + a_i^\dagger b_j)(\langle b_j + a_i^\dagger \rangle) + 4(a_i + b_j^\dagger)(b_j + a_i^\dagger) \langle b_j^\dagger(b_j + a_i^\dagger) \rangle 
\right]
$$

$$
+ (b_j^\dagger + a_i^\dagger)^2 \langle a_i b_j^\dagger \rangle + (b_j + a_i^\dagger)^2 \langle a_i^\dagger b_j^\dagger \rangle .
$$

(C11)

Next we consider $\Delta H_\delta$. Here we can eliminate any terms which involve local averages (e. g. $\langle a_i a_i \rangle$) because they multiply a function whose Fourier coefficient vanishes at zero wave vector. Thereby we have

$$
\Delta H_\delta = \delta J \sum_{(ij)} \sigma_\delta \left[ -\frac{1}{2} a_i a_i (a_i^\dagger b_j^\dagger + b_j) - \frac{1}{2} a_i^\dagger a_i^\dagger (a_i b_j^\dagger + b_j) - \frac{1}{2} b_j^\dagger b_j \langle a_i^\dagger + a_i \rangle 
\right]
$$

$$
- \frac{1}{4} b_j b_j \langle (a_i^\dagger + a_i) b_j^\dagger \rangle + a_i b_j^\dagger (a_i^\dagger b_j) + a_i^\dagger b_j (a_i b_j^\dagger) + b_j a_i (a_i^\dagger b_j^\dagger) + b_j a_i \langle b_j^\dagger a_i^\dagger \rangle \right] .
$$

(C12)

Taking the Hermitian part of this we get

$$
\Delta H_\delta = \delta J \sum_{(ij)} \sigma_\delta \left[ -\frac{1}{2} a_i a_i (a_i^\dagger b_j^\dagger + b_j) - \frac{1}{2} (a_i a_i^\dagger + a_i a_i) \langle a_i(b_j^\dagger + b_j) \rangle - \frac{1}{2} b_j b_j \langle a_i^\dagger + a_i \rangle 
\right]
$$

$$
- \frac{1}{4} (b_j^\dagger b_j + b_j b_j) \langle (a_i^\dagger + a_i) b_j^\dagger \rangle + (a_i^\dagger b_j + a_i b_j^\dagger) \langle b_j^\dagger a_i^\dagger \rangle + (a_i^\dagger b_j^\dagger + a_i b_j) \langle a_i b_j \rangle \right] .
$$

(C13)

Thus we need the averages

$$
X_1 \equiv \langle b_j^2 \rangle = \langle (b_j^\dagger)^2 \rangle = \langle a_i^2 \rangle = \langle (a_i^\dagger)^2 \rangle
$$

(C14a)

$$
X_2 \equiv \langle b_j^\dagger b_j \rangle = \langle a_i a_i^\dagger \rangle
$$

(C14b)

$$
Y_{ij} \equiv \langle b_j a_i^\dagger \rangle = \langle b_j^\dagger a_i \rangle \equiv Y_0 + \sigma_\delta Y
$$

(C14c)

$$
Z_{ij} \equiv \langle b_j^\dagger a_i^\dagger \rangle = \langle b_j a_i \rangle \equiv Z_0 + \sigma_\delta Z ,
$$

(C14d)

where

$$
Y_0 = \frac{1}{4} \sum_j Y_{ij} = \frac{1}{4} \sum_j \langle b_j^\dagger a_i \rangle
$$

(C15a)

$$
Y = \frac{1}{4} \sum_j \sigma_\delta Y_{ij} = \frac{1}{4} \sum_j \sigma_\delta \langle b_j^\dagger a_i \rangle
$$

(C15b)
\[
Z_0 = \frac{1}{4} \sum_j Z_{ij} = \frac{1}{4} \sum_j \langle b_j a_i \rangle \\
Z = \frac{1}{4} \sum_j \sigma_\delta Z_{ij} = \frac{1}{4} \sum_j \sigma_\delta \langle b_j a_i \rangle ,
\]

where the sums over \( j \) are restricted to sites that are nearest neighbors of site \( i \). Now drop terms which sum to zero because of \( \sigma_\delta \) and also those (like \( \sum_{ij} \sigma_\delta a_i^\dagger b_j \)) which do not contribute at zero wavevector. Then we get

\[
\Delta H_J + \Delta H_\delta = -\frac{1}{4} J \sum_{\langle ij \rangle} \left[ (2X_1 + 4Y_0)(a_i a_j^\dagger + a_j a_i^\dagger) + Y_0(b_j^\dagger)^2 + b_j^2 - a_i^2 + (a_i^\dagger)^2 \right] \\
+ 4(X_2 + Z_0)(a_j + b_j)(a_i + b_i^\dagger) \\
+ \frac{1}{8} \delta J I \sum_{ij} \left[ (Y + Z)[-4a_i a_j^\dagger b_j - a_i b_j^\dagger - a_j b_j^\dagger - (a_i^\dagger)^2 - b_j^2 - (b_j^\dagger)^2] \\
+ 8Y(a_i^\dagger b_j + a_j b_i^\dagger) + 8Z(a_i^\dagger b_j + a_i b_j^\dagger) \right].
\]

The coefficients can be evaluated straightforwardly. For instance, if one considers \( \mathcal{H}_J \) as the unperturbed Hamiltonian and treats \( \mathcal{H}_\delta \) as a perturbation, then one has

\[
Y = \frac{1}{4} \sum_j \sigma_\delta \langle b_j^\dagger a_i \rangle \\
= \sum_j \sigma_\delta \left[ \langle 0|b_j^\dagger a_i \frac{1}{\mathcal{E}} \mathcal{H}_\delta |0 \rangle + \langle 0|\mathcal{H}_\delta \frac{1}{\mathcal{E}} b_j^\dagger a_i |0 \rangle \right] ,
\]

where \(|0>\) is the spin-wave vacuum and \( \mathcal{E} \) is the unperturbed energy of the virtual state. We give the evaluations

\[
X_1 = 4C_{2a}(\delta J_1 / J)^2 \\
Y_0 = 4C_{2d}(\delta J_1 / J)^2 \\
Y = -8C_{2a}(\delta J_1 / J) \\
Z = -8C_{2b}(\delta J_1 / J) ,
\]

where

\[
C_{2a} = \frac{1}{128N} \sum_q \frac{|c_x(q) - c_y(q)|^2}{\epsilon(q)^3}
\]

\]
\[ C_{2b} = \frac{1}{128N} \sum_{q} \frac{|c_x(q) - c_y(q)|^2}{\epsilon(q)^3} \gamma(q)^2 \]  
(C19b)

\[ C_{2c} = \frac{1}{128N} \sum_{q} \frac{|c_x(q) - c_y(q)|^2}{\epsilon(q)^5} [1 + 2\gamma(q)^2] \]  
(C19c)

\[ C_{2d} = C_2 - C_{2c} , \]  
(C19d)

where \( \epsilon(q)^2 = 1 - \gamma(q)^2 \), \( C_2 = C_{2a} + C_{2b} \).

To summarize: the effect of quantum fluctuations of the in-plane exchange anisotropy are contained in the effective Hamiltonian of Eq. (C16). Since the result is given in real space, we can apply it now to the 2342 structure where it gives rise to contributions to the dynamical matrices written in Eq. (69). The terms proportional to \( X_2 + Z_0 \) are taken into account by the spin-wave renormalization incorporated in \( Z_c \).

**APPENDIX D: IN–PLANE ANISOTROPIC I–II INTERACTION**

We start from Eq. (71), which can be written as

\[ V_{12} = -\delta J_{12} \sqrt{S/2} \sum_i e_i^\dagger e_i \left( a_{i+x} + a_{i-x}^\dagger + d_{i-x} + d_{i+x}^\dagger - b_{i-y}^\dagger - b_{i+y} - c_{i+y} - c_{i+y}^\dagger \right) \]

\[ +\delta J_{12} \sqrt{S/2} \sum_i f_i^\dagger f_i \left( a_{i-x} + a_{i-x}^\dagger + d_{i+x} + d_{i+x}^\dagger - b_{i+y}^\dagger - b_{i-y} - c_{i-y} - c_{i-y}^\dagger \right) \]

\[ -4\delta J_{12} \sqrt{S/2} \sum_i [e_i^\dagger + e_i - \frac{e_i^\dagger e_i}{2S}] - 4\delta J_{12} \sqrt{S/2} \sum_i [f_i^\dagger + f_i - \frac{f_i^\dagger f_i}{2S}] \]

\[ +\delta J_{12} \sqrt{S/2} \sum_i (e_i^\dagger + e_i)[a_{i+x} a_{i+x}^\dagger + d_{i-x} d_{i-x}^\dagger + b_{i-y} b_{i-y}^\dagger + c_{i+y} c_{i+y}^\dagger] \]

\[ +\delta J_{12} \sqrt{S/2} \sum_i (f_i^\dagger + f_i)[a_{i-x} a_{i-x}^\dagger + d_{i+x} d_{i+x}^\dagger + b_{i+y} b_{i+y}^\dagger + c_{i-y} c_{i-y}^\dagger] . \]  
(D1)

Eliminate terms linear in the boson operators by the shifts

\[ e_i \rightarrow e_i + s , \quad f_i \rightarrow f_i + s , \]

\[ a_i \rightarrow a_i + t , \quad b_i \rightarrow b_i + t , \quad c_i \rightarrow c_i + t , \quad d_i \rightarrow d_i + t . \]  
(D2)

The corresponding Fourier transforms are shifted by a factor \( \sqrt{N_{uc}} \).

\[ \sum_i e_i = e(0) \rightarrow e(0) + \sqrt{N_{uc}} s , \quad \sum_i a_i = a(0) \rightarrow a(0) + \sqrt{N_{uc}} t . \]  
(D3)

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In what follows $e$ will denote $e(q = 0)$ and similarly for other operators. Then the linear terms in the Hamiltonian are

$$V_1 = -4\delta J_{12} S \sqrt{N_{\text{wc}} S/2} \left(e^\dagger + e + f^\dagger + f\right). \tag{D4}$$

The quadratic terms in the isotropic part of the Hamiltonian are

$$\begin{aligned}
V_2 &= (4J + 2J_3) S (a^\dagger a + b^\dagger b + c^\dagger c + d^\dagger d) \\
&+ 4J_2 S (e^\dagger e + f^\dagger f) + J_{12} S \left([a^\dagger + d^\dagger] f + [b^\dagger + c^\dagger] e + [a + d] f^\dagger + [b + c] e^\dagger\right) \\
&+ (2J + 2J_3) S (a^\dagger b^\dagger c^\dagger d^\dagger + ab + cd) + 2JS (a^\dagger c^\dagger + b^\dagger d^\dagger + ac + bd) + 4J_2 S (e^\dagger f^\dagger + ef) \\
&+ J_{12} S \left([a^\dagger + d^\dagger] e^\dagger + [b^\dagger + c^\dagger] f^\dagger + [a + d] e^\dagger + [b + c] f^\dagger\right). \tag{D5}
\end{aligned}$$

Now

$$\begin{aligned}
\frac{\partial (V_1 + V_2)}{\partial e} &= 0 = \sqrt{N_{\text{wc}} S} \left(-4\delta J_{12} \sqrt{S/2} + 2J_{12} t + 4J_2 s + 2J_{12} t + 4J_2 s\right) \\
\frac{\partial (V_1 + V_2)}{\partial a} &= 0 = \sqrt{N_{\text{wc}} S} \left((4J + 2J_3) t + J_{12} s + (2J + 2J_3) t + 2J t + J_{12} s\right). \tag{D6}
\end{aligned}$$

For $J_{12}^2 \ll 4JJ_2$ we have

$$\begin{aligned}
s &= \frac{4\delta J_{12} \sqrt{S/2}}{8J_2}, \tag{D7} \\
t &= \frac{2J_{12} s}{8J + 4J_3} = -\frac{J_{12} \delta J_{12} \sqrt{S/2}}{J_2 (8J + 4J_3)}. \tag{D8}
\end{aligned}$$

As discussed in the text, these are the expected results.

Now we record the terms in the Hamiltonian which are cubic in boson operators:

$$\begin{aligned}
\mathcal{H}^{(3)} &= \delta J_{12} \sqrt{S/2} \left\{ -\sum_{i \in e} e_i^\dagger e_i \left(a^\dagger_{i+x} + a_{i+x} + d^\dagger_{i-x} + d_{i-x} - b^\dagger_{i-y} - b_{i-y} - c^\dagger_{i+y} - c_{i+y}\right) \\
&+ \sum_{i \in f} f_i^\dagger f_i \left(b^\dagger_{i-x} + b_{i-x} + a^\dagger_{i+x} + a_{i+x} - c^\dagger_{i+y} - c_{i+y} - b^\dagger_{i-y} - b_{i-y}\right) \\
&+ \sum_{i \in e} e_i^\dagger e_i e_i + \sum_{i \in f} f_i^\dagger f_i f_i \\
&+ \sum_{i \in e} (e_i^\dagger + e_i) \left(a^\dagger_{i+x} a_{i+x} + d^\dagger_{i-x} d_{i-x} + b^\dagger_{i-y} b_{i-y} + c^\dagger_{i+y} c_{i+y}\right) \\
&+ \sum_{i \in f} (f_i^\dagger + f_i) \left(b^\dagger_{i-x} b_{i-x} + a^\dagger_{i+x} a_{i+x} + b^\dagger_{i+y} b_{i+y} + c^\dagger_{i+y} c_{i-y}\right) \right\}. \tag{D9}
\end{aligned}$$
Now make the replacements of Eq. (D3) to get

\[
\mathcal{H}^{(3)} = \langle e \rangle \delta J_{12} \sqrt{S/2} \left\{ - \sum_{i \in e} (e_i^1 + e_i) \left( a_i^1 + a_{i+x} + a_{i+y} + d_i^1 - b_i^1 - c_i^1 \right) \\
+ \sum_{i \in f} (f_i^1 + f_i) \left( a_i^1 + a_{i-x} + a_{i-y} + d_i^1 - b_i^1 - c_i^1 \right) \\
+ \frac{1}{2} \sum_{i \in e} \left| e_i^1 \right|^2 + \frac{1}{2} \sum_{i \in f} \left| f_i \right|^2 + \mathcal{O}(1) \right\}
\]

Here we dropped the terms proportional to \( \langle a \rangle \). They are smaller than those in \( \langle e \rangle \) by \( J_{12}/(4J) \approx 1/50 \). Also, as before, to this order in \( 1/S \) we may replace the perturbation by its Hermitian part. Then the effective quadratic terms above are

\[
\mathcal{H}^{(3)} = \frac{\left( \delta J_{12} \right)^2 S}{4J_2} \left\{ - \sum_{i \in e} (e_i^1 + e_i) \left( a_i^1 + a_{i+x} + a_{i+y} + d_i^1 - b_i^1 - c_i^1 \right) \\
+ \sum_{i \in f} (f_i^1 + f_i) \left( a_i^1 + a_{i-x} + a_{i-y} + d_i^1 - b_i^1 - c_i^1 \right) \\
+ \frac{1}{2} \sum_{i \in e} \left| e_i^1 \right|^2 + \frac{1}{2} \sum_{i \in f} \left| f_i \right|^2 + \mathcal{O}(1) \right\}
\]

which leads to Eq. (74).

Now we look at the fourth order terms in the CuII – CuII isotropic exchange interaction.

These are

\[
V_{DM} = -\frac{1}{2} J_2 \sum_{i \in e, \delta} \left( e_i^1 f_{i+\delta}^1 f_{i+\delta} f_{i+\delta} + f_{i+\delta} e_i^1 e_i^1 + 2 e_i^1 e_i^1 f_{i+\delta}^1 f_{i+\delta} \right).
\]

(D12)
Substituting in two shifts of \( \langle e \rangle \), this is

\[
V_{DM} = -\frac{1}{2} \langle e \rangle^2 J_2 \sum_{i, \delta} \left[ e_i^\dagger f_{i, \delta} + 2 f_{i, \delta}^\dagger e_i + 2 e_i^\dagger f_{i, \delta}^\dagger + (f_{i, \delta}^\dagger)^2 + 2 f_{i, \delta} e_i \\
+ f_{i, \delta}^\dagger e_i^\dagger + 2 e_i^\dagger e_i + 2 f_{i, \delta}^\dagger f_{i, \delta} + 2 (e_i^\dagger + e_i)(f_{i, \delta}^\dagger + f_{i, \delta}) \right].
\]

(D13)

Taking the Hermitian part of this, we get

\[
V_{DM} = -\frac{(\delta J_{12})^2 S}{16 J_2} \sum_{i, \delta} \left[ e_i^\dagger f_{i, \delta} + e_i f_{i, \delta}^\dagger + 2 e_i^\dagger f_{i, \delta}^\dagger + 2 e_i f_{i, \delta} + 4 f_{i, \delta}^\dagger f_{i, \delta} \\
+ 4 e_i^\dagger e_i + \frac{1}{2} e_i^2 + \frac{1}{2} (e_i^\dagger)^2 + \frac{1}{2} f_{i, \delta}^2 + \frac{1}{2} (f_{i, \delta}^\dagger)^2 + 2 (e_i^\dagger + e_i)(f_{i, \delta}^\dagger + f_{i, \delta}) \right]
\]

\[
= -\frac{(\delta J_{12})^2 S}{16 J_2} \sum_q \left[ \left( 6 e(q) f^\dagger(q) + 6 f^\dagger(q) e(q) + 8 e(q) f(-q) + 8 f^\dagger(q) f^\dagger(-q) \right)(e_x + e_y) \\
+ 16 f^\dagger(q) f(q) + 16 e^\dagger(q) e(q) + 2 e^\dagger(q) e^\dagger(-q) + 2 e(q) e(-q) \\
+ 2 f^\dagger(q) f^\dagger(-q) + 2 f(q) f(-q) \right],
\]

(D14)

which leads to Eq. (75).

Contributions from quartic terms in the CuI – CuII interaction are smaller, i.e. of order, \((\delta J_{12})^2 J_{12}^2/(J J_2^2)\), if we take out one factor of \( \langle e \rangle \) and one factor of \( \langle a \rangle \). Taking out two \( \langle a \rangle \) factors gives an even smaller result. Taking out two \( \langle a \rangle \) shifts from the CuI – CuI anharmonic term gives a contribution of order \( J_{12}^2 \delta J_{12}^2/(J J_2)^2 \). All these terms are neglected.

**APPENDIX E: IN – PLANE ANISOTROPIC II – II INTERACTION**

1. **Self-Energy Due to Cubic Perturbations**

   We start by discussing how one constructs the self-energy due to cubic perturbations. The point is that we wish to avoid the complexities involving Matsubara sums etc. Let us suppose that we have an unperturbed Hamiltonian in terms of normal mode operators, \( E(q) \) and \( F(q) \):

\[
\mathcal{H} = \sum_q \omega(q) \left( E^\dagger(q) E(q) + F^\dagger(q) F(q) \right).
\]

(E1)

Now we want to identify the perturbative contributions to the matrices \( A(q) \) and \( B(q) \). Suppose we wish to calculate perturbative contributions leading to an effective quadratic Hamiltonian of the form

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\[ \frac{1}{2} B(q) E^\dagger(q) E^\dagger(-q). \] (E2)

For this purpose we make the identification
\[ \delta B(q) = \langle 0 | E(q) E(-q) V \frac{1}{\xi} V | 0 \rangle. \] (E3)

Thus for \( \omega(q) \to 0 \) and considering only the ground state, we may write
\[ \delta B(q) = \langle 0 | \partial V / \partial E^\dagger(q) \frac{1}{\xi} \partial V / \partial E^\dagger(-q) | 0 \rangle \]
\[ + \langle 0 | \partial V / \partial E^\dagger(-q) \frac{1}{\xi} \partial V / \partial E^\dagger(q) | 0 \rangle. \] (E4)

Similarly for the term in the Hamiltonian
\[ A(q) E^\dagger(q) F(q) \] (E5)

we make the identification
\[ \delta A(q) = \langle 0 | E(q) V \frac{1}{\xi} V F^\dagger(q) | 0 \rangle. \] (E6)

Thus for \( \omega(q) \to 0 \) and considering only the ground state, we may write
\[ \delta A(q) = \langle 0 | \partial V / \partial E^\dagger(q) \frac{1}{\xi} \partial V / \partial F(q) | 0 \rangle \]
\[ + \langle 0 | \partial V / \partial F(q) \frac{1}{\xi} \partial V / \partial E^\dagger(q) | 0 \rangle. \] (E7)

This type of relation holds generally under the two assumptions: a) we consider the perturbation to modes whose energy can be neglected in the energy denominators, and b) we consider only the ground and low lying excited states, so that boson occupation numbers are zero.

We have made the identification in terms of the normal mode operators, but equally we may transform to any set of modes.

2. Application to CuII - CuII In-Plane Interactions

We start from Eq. (76) and implement the results of the preceding subsection. For small \( k \) we write
\[ T_1 \equiv \partial V/\partial e(k) = \delta J_2 \sqrt{2S/N_{uc}} \sum_q [f(q) + f^\dagger(-q)]e^\dagger(q)(c_x - c_y) \]

\[ T_2 \equiv \partial V/\partial e^\dagger(k) = \delta J_2 \sqrt{2S/N_{uc}} \sum_q [f(q) + f^\dagger(-q)]e(-q)(c_x - c_y) \]

\[ T_3 \equiv \partial V/\partial f(k) = -\delta J_2 \sqrt{2S/N_{uc}} \sum_q [e(q) + e^\dagger(-q)]f^\dagger(q)(c_x - c_y) \]

\[ T_4 \equiv \partial V/\partial f^\dagger(k) = -\delta J_2 \sqrt{2S/N_{uc}} \sum_q [e(q) + e^\dagger(-q)]f(-q)(c_x - c_y) , \]

(E8)

where \( c_x = \cos(aq_x) \) and \( c_y = \cos(aq_y) \). Thus if \( \mathbf{\tilde{p}} \) denotes \( -\mathbf{p} \), then

\[
\langle T_1 \frac{1}{E} T_1 \rangle = \frac{2S(\delta J_2)^2}{N_{uc}} \sum_{q,p} \left\langle \frac{1}{E} [f(q) + f^\dagger(q)] e^\dagger(q)(c_x - c_y) \right\rangle \]

\[
= \frac{2S(\delta J_2)^2}{N_{uc}} \sum_{q,p} \left\langle \frac{1}{E} \left[ -m_p E_p + l_p f^\dagger l_p f^\dagger \right] e^\dagger(q)(c_x - c_y) \right\rangle \]

\[
= -\frac{2S(\delta J_2)^2}{N_{uc}} \sum_q \frac{(c_x - c_y)^2 l_p^2 m_p^2}{8J_2 S\epsilon(q)} = -\frac{(\delta J_2)^2}{4J_2 N_{uc}} \sum_q \frac{(c_x - c_y)^2}{\epsilon(q)^3} \frac{\gamma(q)^2}{4} , \]

(E9)

where we introduced the normal mode operators

\[
E_q = l_q e(q) - m_q f^\dagger(q) ,
\]

\[
F_q = l_q f(q) - m_q e^\dagger(q) , \]

(E10)

where

\[
l_q^2 = \frac{1 + \epsilon(q)}{2\epsilon(q)} , \quad m_q^2 = \frac{1 - \epsilon(q)}{2\epsilon(q)} , \quad l_q m_q = -\frac{\gamma(q)}{2\epsilon(q)} , \]

(E11)

where \( \gamma(q) = \frac{1}{2}[\cos(aq_x) + \cos(aq_y)] \). Similarly

\[
\langle T_1 \frac{1}{E} T_2 \rangle = \langle T_2 \frac{1}{E} T_1 \rangle = -\frac{(\delta J_2)^2}{4J_2 N_{uc}} \sum_q \frac{(c_x - c_y)^2}{\epsilon(q)^3} \left( \frac{1}{2} + \frac{\gamma(q)^2}{4} \right) \]

(E12)

\[
\langle T_1 \frac{1}{E} T_3 \rangle = \langle T_3 \frac{1}{E} T_1 \rangle = \frac{(\delta J_2)^2}{4J_2 N_{uc}} \sum_q \frac{(c_x - c_y)^2}{\epsilon(q)^3} \left( \frac{3}{4} \gamma(q)^2 \right) \]

(E13)

\[
\langle T_1 \frac{1}{E} T_4 \rangle = \langle T_4 \frac{1}{E} T_1 \rangle = \frac{(\delta J_2)^2}{4J_2 N_{uc}} \sum_q \frac{(c_x - c_y)^2}{\epsilon(q)^3} \left( \frac{1 - \epsilon(q)^2}{4} \right) . \]

(E14)

Now we have the contribution to the coefficient of \( e^\dagger e \), which we denote \( \delta a_{55} \), as

\[
\delta a_{55} = \langle T_1 \frac{1}{E} T_2 \rangle + \langle T_2 \frac{1}{E} T_1 \rangle = \left[ \delta J_2^2 / J_2 \right] [-32C_{2a} - 16C_{2b}] , \]

(E15)

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where $C_{2a}$ and $C_{2b}$ were defined in Eq. (C19).

Likewise the contribution to the coefficient of $e^\dagger f$ which we denote $\delta a_{56}$ is

$$
\delta a_{56} = \langle T_1 \frac{1}{e} T_4 \rangle + \langle T_4 \frac{1}{e} T_1 \rangle = \left[ (\delta J_2)^2 / J_2 \right] [32C_{2a} - 16C_{2b}] . \tag{E16}
$$

Similarly, $\delta b_5$ is the contribution to the coefficient of $\frac{1}{e} e^\dagger e^\dagger$, so that

$$
\delta b_{55} = 2 \langle T_1 \frac{1}{e} T_1 \rangle = \left[ (\delta J_2)^2 / J_2 \right] [-16C_{2b}] \tag{E17}
$$

and likewise

$$
\delta b_{55} = \langle T_1 \frac{1}{e} T_3 \rangle + \langle T_3 \frac{1}{e} T_1 \rangle = \left[ (\delta J_2)^2 / J_2 \right] [48C_{2b}] . \tag{E18}
$$
APPENDIX F: INTERPLANAR ANISOTROPIC CUII - CUII INTERACTION

1. Pseudodipolar Interactions

In order to facilitate the evaluation of the lattice sums we parametrize the anisotropic exchange interactions between the \(i\)th CuII spin in one plane and the nearest neighboring \(j\)th CuII spin in an adjacent layer. We introduce the indicator variable \(\sigma_i\) which is unity if \(i\) is on the \(e\)-sublattice and is \(-1\) if \(i\) is on the \(f\)-sublattice. We also introduce a variable \(\mu_i\) to distinguish between the two nearest neighboring sites with the same value of \(\sigma_i\). Then for the interaction between nearest neighboring CuII spins \(i\) and \(j\) in adjacent CuO layers we use Fig. 4 to write the principal axes as

\[
\hat{n}^{(ij)}_1 = \left[ \frac{1}{2} (1 + \sigma_i \sigma_j) \hat{\eta} - \frac{1}{2} (1 - \sigma_i \sigma_j) \hat{\xi} \right] \mu_i \mu_j \tag{F1a}
\]

\[
\hat{n}^{(ij)}_2 = \left[ \frac{1}{2} (1 + \sigma_i \sigma_j) \hat{\xi} \cos \psi + \frac{1}{2} (1 - \sigma_i \sigma_j) \hat{\eta} \cos \psi \right] \mu_i \mu_j + \hat{z} \sin \psi \tag{F1b}
\]

\[
\hat{n}^{(ij)}_3 = \left[ \frac{1}{2} (1 + \sigma_i \sigma_j) \hat{\xi} \sin \psi + \frac{1}{2} (1 - \sigma_i \sigma_j) \hat{\eta} \sin \psi \right] \mu_i \mu_j - \hat{z} \cos \psi . \tag{F1c}
\]

We also write

\[
S_i = -\sigma_i (S - a_i \dagger a_i) \hat{\xi} + \sqrt{S/2}(a_i \dagger + a_i) \hat{\eta} + i\sqrt{S/2}(a_i - a_i \dagger) \hat{z} \sigma_i , \tag{F2}
\]

where, in this appendix, \(a_i\) is the boson operator for spin \(i\). Then we have

\[
S_i \cdot \hat{n}^{(ij)}_1 = \frac{1}{2} (\sigma_i - \sigma_j) \mu_i \mu_j (S - a_i \dagger a_i) + \frac{1}{2} (1 + \sigma_i \sigma_j) \mu_i \mu_j \sqrt{S/2}(a_i \dagger + a_i) \tag{F3a}
\]

\[
S_i \cdot \hat{n}^{(ij)}_2 = -\frac{1}{2} (S - a_i \dagger a_i) c(\sigma_i + \sigma_j) \mu_i \mu_j + \frac{1}{2} (1 - \sigma_i \sigma_j) \mu_i \mu_j \sqrt{S/2}(a_i \dagger + a_i) + i|\sigma_i| \sqrt{S/2}(a_i - a_i \dagger) \tag{F3b}
\]

\[
S_i \cdot \hat{n}^{(ij)}_3 = -\frac{1}{2} (S - a_i \dagger a_i) s(\sigma_i + \sigma_j) \mu_i \mu_j + \frac{1}{2} (1 - \sigma_i \sigma_j) \mu_i \mu_j \sqrt{S/2}(a_i \dagger + a_i) - i|\sigma_i| \sqrt{S/2}(a_i - a_i \dagger) , \tag{F3c}
\]

where \(c \equiv \cos \psi\) and \(s \equiv \sin \psi\). Then we have

\[
H_{ij} = S \sum_{m=1}^{3} K_m [S_i \cdot \hat{n}_m][S_j \cdot \hat{n}_m] \equiv S \sum_{m=1}^{3} K_m T_m , \tag{F4}
\]

where, at quadratic order,

\[
T_1 = \frac{1}{2} (1 - \sigma_i \sigma_j) (a_i \dagger a_i + a_j \dagger a_j) + \frac{1}{2} (1 + \sigma_i \sigma_j) (a_i \dagger + a_i)(a_j \dagger + a_j)
\]

\[
T_2 = -\frac{1}{2} (1 + \sigma_i \sigma_j) c^2 (a_i \dagger a_i + a_j \dagger a_j) + \frac{1}{2} \left( \frac{3}{2} (1 - \sigma_i \sigma_j) (a_i \dagger + a_i)c\mu_i \mu_j + i|\sigma_i| s(a_i - a_i \dagger) \right) \times \left( \frac{3}{2} (1 - \sigma_i \sigma_j) (a_j \dagger + a_j)c\mu_i \mu_j + i|\sigma_j| s(a_j - a_j \dagger) \right) \tag{F5}
\]

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and $T_3$ is obtained from $T_2$ by replacing $\sin \psi$ by $-\cos \psi$ and $\cos \psi$ by $\sin \psi$. Thereby we get the diagonal contribution to the Hamiltonian as

$$\delta \mathcal{H} = \delta V_{i,i+\delta} = 4\Delta KS \sum_i a_i^\dagger a_i,$$

where $\Delta K$ was defined in Eq. (83).

The remaining contributions to the Hamiltonian are found from Eq. (F4) to be

$$\delta \mathcal{H} = \frac{1}{2} \sum_{i \in \Pi, j} \left\{ \frac{1}{4} K_1 S (1 + \sigma_i \sigma_j)(a_i^\dagger + a_i)(a_j^\dagger + a_j)ight.\
+ \frac{1}{2} K_2 S \left[ \frac{1}{2} c^2 (1 - \sigma_i \sigma_j)(a_i^\dagger + a_i)(a_j^\dagger + a_j) - \sigma_i \sigma_j s^2 (a_i - a_i^\dagger)(a_j - a_j^\dagger) \\
+ ic \mu_i \mu_j (\sigma_j - \sigma_i)(a_i^\dagger + a_i)(a_j - a_j^\dagger) + \ldots \right\},$$

where $\ldots$ indicates further terms in $K_3$ obtained from those of $K_2$ by replacing $\cos \psi$ by $\sin \psi$ and $\sin \psi$ by $-\cos \psi$ and $j$ is summed over CuII nearest neighbors in adjacent planes. For $q_x = q_y = 0$ the imaginary term gives zero contribution to the dynamical matrices. Then, the terms with $\sigma_i = \sigma_j$ give a contribution to the Hamiltonian of

$$\delta \mathcal{H} = \frac{S}{4} \sum_i \sum_{j: \sigma_j = \sigma_i} \left[ K_1 (a_i^\dagger + a_i)(a_j^\dagger + a_j) - K_2 s^2 (a_i - a_i^\dagger)(a_j - a_j^\dagger) \\
- K_3 s^2 (a_i - a_i^\dagger)(a_j - a_j^\dagger) \right].$$

The terms with $\sigma_i = -\sigma_j$ give a contribution to the Hamiltonian of

$$\delta \mathcal{H} = \frac{S}{4} \sum_i \sum_{j: \sigma_j = -\sigma_i} \left[ K_2 c^2 (a_i^\dagger + a_i)(a_j^\dagger + a_j) + K_2 s^2 (a_i - a_i^\dagger)(a_j - a_j^\dagger) \\
+ K_3 s^2 (a_i^\dagger + a_i)(a_j^\dagger + a_j) + K_3 c^2 (a_i - a_i^\dagger)(a_j - a_j^\dagger) \right].$$

The term in Eq. (F6) and the number conserving terms in Eq. (F8) reproduce Eq. (82a) and the other terms in Eq. (F8) reproduce Eq. (82c). Equation (F9) reproduces Eqs. (82b) and (82d).

2. Dipolar Interactions

For the dipolar interactions it is convenient to construct the Hamiltonian explicitly rather than to identify it with the pseudodipolar interaction. We substitute Eq. (F2) into the dipolar
interaction to get

\[ \mathcal{H}_{ij} = g^2 \mu_B^2 r_{ij}^{-3} [ \mathbf{S}_i \cdot \mathbf{S}_j - 3 ( \mathbf{S}_i \cdot \mathbf{r}_{ij} ) ( \mathbf{S}_j \cdot \mathbf{r}_{ij} ) ] \rightarrow -3g^2 \mu_B^2 r_{ij}^{-3} ( \mathbf{S}_i \cdot \mathbf{r}_{ij} ) ( \mathbf{S}_j \cdot \mathbf{r}_{ij} ) \]

\[ = - \frac{3g^2 \mu_B^2}{r_{ij}^3} \left[ -\sigma_i [ S - a_i^1 a_i^1 ] ( \xi \cdot \mathbf{r}_{ij} ) + \sqrt{2} ( a_i^1 + a_i^1 ) ( \eta \cdot \mathbf{r}_{ij} ) + i \sigma_i \sqrt{2} ( a_i^1 - a_i^1 ) ( \xi \cdot \mathbf{r}_{ij} ) \right] \times \left[ -\sigma_j [ S - a_j^1 a_j^1 ] ( \xi \cdot \mathbf{r}_{ij} ) + \sqrt{2} ( a_j^1 + a_j^1 ) ( \eta \cdot \mathbf{r}_{ij} ) + i \sigma_j \sqrt{2} ( a_j^1 - a_j^1 ) ( \xi \cdot \mathbf{r}_{ij} ) \right]. \]  

(F10)

Here we dropped the term in \( \mathbf{S}_i \cdot \mathbf{S}_j \) which may be included in the isotropic Heisenberg Hamiltonian. At quadratic order this gives

\[ \mathcal{H} = \frac{1}{2} \sum_{i,j \in I} \mathcal{H}_{ij} = \sum_{i,j \in I} \frac{3g^2 \mu_B^2 S}{2r_{ij}^3} \left[ \sigma_i \sigma_j ( a_i^1 a_j^1 + a_j^1 a_i^1 ) ( \xi \cdot \mathbf{r}_{ij} )^2 - \frac{1}{2} ( a_i + a_i^1 ) ( a_j + a_j^1 ) ( \eta \cdot \mathbf{r}_{ij} )^2 \right. \\
\left. + \frac{1}{2} \sigma_i \sigma_j ( a_i - a_i^1 ) ( a_j - a_j^1 ) ( \xi \cdot \mathbf{r}_{ij} )^2 - i \sigma_j ( a_i + a_i^1 ) ( a_j - a_j^1 ) ( \xi \cdot \mathbf{r}_{ij} ) ( \eta \cdot \mathbf{r}_{ij} ) \right]. \]  

(F11)

We now consider what contributions this gives to the dynamical matrix for \( q_x = q_y = 0 \). Then the imaginary term can be dropped. For simplicity we truncate the sums to include only interactions between adjacent planes. Then we have

\[ \delta a_{55} = \sum_{j \in c} \frac{3g^2 \mu_B^2 S}{r_{ij}^3} \left[ ( \xi \cdot \mathbf{r}_{ij} )^2 - \frac{1}{2} c_z ( \eta \cdot \mathbf{r}_{ij} )^2 - \frac{1}{2} c_z ( \xi \cdot \mathbf{r}_{ij} )^2 \right] + \sum_{j \in f} \frac{3g^2 \mu_B^2 S}{r_{ij}^3} \sigma_j ( \xi \cdot \mathbf{r}_{ij} )^2 \]  

(F12a)

\[ \delta a_{56} = \sum_{j \in f} \frac{3g^2 \mu_B^2 S}{r_{ij}^3} \left[ -\frac{1}{2} ( \eta \cdot \mathbf{r}_{ij} )^2 + \frac{1}{2} ( \xi \cdot \mathbf{r}_{ij} )^2 \right] c_z \]  

(F12b)

\[ \delta b_{55} = \sum_{j \in e} \frac{3g^2 \mu_B^2 S}{r_{ij}^3} \left[ -\frac{1}{2} ( \eta \cdot \mathbf{r}_{ij} )^2 + \frac{1}{2} ( \xi \cdot \mathbf{r}_{ij} )^2 \right] c_z \]  

(F12c)

\[ \delta b_{56} = \sum_{j \in f} \frac{3g^2 \mu_B^2 S}{r_{ij}^3} \left[ -\frac{1}{2} ( \eta \cdot \mathbf{r}_{ij} )^2 + \frac{1}{2} ( \xi \cdot \mathbf{r}_{ij} )^2 \right] c_z \]  

(F12d)

where \( c_z = \cos(q_z c/2) \), \( i \) is a fixed site in the \( e \) sublattice, and the sum over \( j \) is restricted to the planes adjacent to site \( i \).

This interaction is negligibly small except with respect to the lowest in-plane mode. So we only need the combination

\[ \delta (a_{55} + b_{55} - a_{56} - b_{56}) = \sum_{j \in I: z_{ij} = \pm c/2} \frac{3g^2 \mu_B^2 S}{r_{ij}^3} \left[ \sigma_j ( \xi \cdot \mathbf{r}_{ij} )^2 - \sigma_j ( \eta \cdot \mathbf{r}_{ij} )^2 c_z \right] \]

\[ = \sum_{j \in I: z_{ij} = c/2} \frac{3g^2 \mu_B^2 S}{r_{ij}^3} \left[ \sigma_j ( x_{ij} + y_{ij} )^2 - \sigma_j c_z ( x_{ij} - y_{ij} )^2 \right]. \]  

(F13)
Note that the sum over sites \( j \) in an adjacent plane from site \( i \) vanishes:

\[
\sum_{j \in \Pi: z_j = c/2} \frac{\sigma_j x_{ij}^2}{r_{ij}^5} = \sum_{j \in \Pi: z_j = c/2} \frac{\sigma_j y_{ij}^2}{r_{ij}^5} = 0.
\]  

(F14)

Thus

\[
\delta(a_{55} + b_{55} - a_{56} - b_{56}) = 6(1 + c_z) g^2 \mu_B S \sum_{j \in \Pi: z_j = c/2} \frac{\sigma_j x_{ij} y_{ij}}{r_{ij}^5}.
\]  

(F15)

**APPENDIX G: INTERPLANAR ANISOTROPIC CUI - CUII INTERACTION**

For the CuI sites we introduce further indicator variables \( \tau \) (which tells the direction of the moment) and \( \rho \) (which discriminates between sublattices) such that \( \tau = \rho = 1 \) for an \( a \) site, \( -\tau = \rho = 1 \) for a \( b \) site, \( \tau = \rho = -1 \) for a \( c \) site, and \( \tau = -\rho = 1 \) for a \( d \) site. Then, from Fig. 5, we have the principal axes for the sites \( i \) and \( j \) where \( i \) (\( j \)) is in the CuI (II) sublattice as

\[
\hat{m}_1^{(ij)} = -\frac{\rho_i \sigma_j}{\sqrt{2}} \left[ \hat{\xi} + \tau_i \hat{\eta} \right] 
\]  

(G1a)

\[
\hat{m}_2^{(ij)} = \hat{\zeta} \cos \phi + \frac{\rho_i \sigma_j \sin \phi}{\sqrt{2}} \left[ \hat{\eta} - \tau_i \hat{\xi} \right] 
\]  

(G1b)

\[
\hat{m}_3^{(ij)} = -\hat{\zeta} \sin \phi + \frac{\rho_i \sigma_j \cos \phi}{\sqrt{2}} \left[ \hat{\eta} - \tau_i \hat{\xi} \right]. 
\]  

(G1c)

In checking the above it is useful to note that changing the sign of either \( \rho_i \) or \( \sigma_j \) induces a 180° rotation about the \( z \)-axis.

Also we use Eq. (F2) for the CuII spins and

\[
S_i = \tau_i (S - a_i^\dagger a_i) \hat{\xi} + \sqrt{S/2} (a_i + a_i^\dagger) \hat{\eta} + i \tau_i \sqrt{S/2} (a_i^\dagger - a_i) \hat{\zeta} 
\]  

(G2)

for the CuI spins. Thus if \( i \) labels a CuI spin we have

\[
\hat{m}_1^{(ij)} \cdot S_i = \frac{\rho_i \tau_i \sigma_j}{\sqrt{2}} \left[ -(S - a_i^\dagger a_i) - \sqrt{S/2} (a_i + a_i^\dagger) \right] 
\]  

(G3a)

\[
\hat{m}_2^{(ij)} \cdot S_i = \frac{1}{\sqrt{2}} \left[ -\rho_i \sigma_j s(S - a_i^\dagger a_i) + \rho_i \sigma_j s \sqrt{S/2} (a_i + a_i^\dagger) + i \tau_i c \sqrt{S} (a_i^\dagger - a_i) \right] 
\]  

(G3b)

\[
\hat{m}_3^{(ij)} \cdot S_i = \frac{1}{\sqrt{2}} \left[ -\rho_i \sigma_j c(S - a_i^\dagger a_i) + \rho_i \sigma_j c \sqrt{S/2} (a_i + a_i^\dagger) - i \tau_i s \sqrt{S} (a_i^\dagger - a_i) \right] 
\]  

(G3c)

and if \( j \) labels a CuII spin we have
\[
\dot{m}_1^{(ij)} \cdot S_j = \frac{\rho_i}{\sqrt{2}} \left[ \left( S - a_{ji} a_j \right) - \tau_i \sigma_j \sqrt{S/2} (a_j + a_{ji}^\dagger) \right]
\]
\[
\dot{m}_2^{(ij)} \cdot S_j = \frac{1}{\sqrt{2}} \left[ \rho_i \tau_i s (S - a_{ji}^\dagger a_j) + \rho_i \sigma_j s \sqrt{S/2} (a_j + a_{ji}^\dagger) + i \sigma_j c \sqrt{S} (a_j - a_{ji}^\dagger) \right]
\]
\[
\dot{m}_3^{(ij)} \cdot S_j = \frac{1}{\sqrt{2}} \left[ \rho_i \tau_i c (S - a_{ji}^\dagger a_j) + \rho_i \sigma_j c \sqrt{S/2} (a_j + a_{ji}^\dagger) - i \sigma_j s \sqrt{S} (a_j - a_{ji}^\dagger) \right],
\]

where \( c \equiv \cos \phi \) and \( s \equiv \sin \phi \). We now write
\[
\mathcal{H}_{ij} = \sum_{m=1}^3 [\dot{m}_m \cdot S_i] \sum_{m=1}^3 [\dot{m}_m \cdot S_j] \equiv S \sum_{m=1}^3 K_m^r T_m,
\]
and at quadratic order we have
\[
T_1 = \frac{1}{2} \tau_i \sigma_j [a_{ji}^\dagger a_i + a_j^\dagger a_j] + \frac{1}{4} [a_i + a_{ji}^\dagger][a_j + a_{ji}^\dagger] \tag{G6a}
\]
\[
T_2 = \frac{1}{2} \tau_i \sigma_j s^2 [a_{ji}^\dagger a_i + a_j^\dagger a_j]
+ \left[ \frac{1}{2} \rho_i \sigma_j s (a_i + a_{ji}^\dagger) + \frac{i \tau_i c}{\sqrt{2}} (a_j^\dagger - a_i) \right] \left[ \frac{1}{2} \rho_i \sigma_j s (a_j + a_{ji}^\dagger) + \frac{i \sigma_j c}{\sqrt{2}} (a_j - a_{ji}^\dagger) \right] \tag{G6b}
\]
\[
T_3 = \frac{1}{2} \tau_i \sigma_j c^2 [a_{ji}^\dagger a_i + a_j^\dagger a_j]
\left[ \frac{1}{2} \rho_i \sigma_j c (a_i + a_{ji}^\dagger) - \frac{i \tau_i s}{\sqrt{2}} (a_j^\dagger - a_i) \right]
+ \left[ \frac{1}{2} \rho_i \sigma_j c (a_j + a_{ji}^\dagger) - \frac{i \sigma_j s}{\sqrt{2}} (a_j - a_{ji}^\dagger) \right]. \tag{G6c}
\]

We drop terms which do not contribute to the dynamical matrix for \( q_x = q_y = 0 \) and thereby find that
\[
\mathcal{H} = \sum_{i \in I, j \in I} \mathcal{H}_{ij} = S \sum_{i \in I, j \in I} \left\{ \frac{1}{4} (a_{ji}^\dagger + a_i)(a_j^\dagger + a_j) \left( K_1^r + K_2^r s^2 + K_3^r c^2 \right)
- \frac{1}{2} (a_i^\dagger - a_i)(a_j - a_{ji}^\dagger)(K_2^r c^2 + K_3^r s^2) \tau_i \sigma_j \right\}.
\]

This result reproduces that of Eq. (90).
REFERENCES


19 R. Coldea, S. M. Hayden, G. Aeppli, T. G. Perring, C. D. Frost, T. E. Mason, S.-W.Cheong, and Z. Fisk, cond/mat 0006384 show that at high energies the spin-wave modes are significantly affected by ring exchange involving four sites. We have not included this interaction in our analysis. At low energy such an interaction would probably amount to a change in $J$ of about 5%.


26 V. Yu. Yushankhai and R. Hayn, cond-mat/9902106.


29 Y. J. Kim (Ph. D. Thesis, Harvard, 2000) used the data for LCO in Ref. 33 and for 2122 in Ref. 34, to estimate $J$ via an interpolation with respect to lattice constant.


31 F. C. Chou, private communication gives.


FIG. 1. Magnetic structure of 2342. The CuI spins (in sublattices $a$, $b$, $c$, and $d$) are thick arrows and the CuII spins (in sublattices $e$ and $f$) are thin arrows. The basis vectors for the magnetic unit cell are $a_1 = a(\hat{x} + \hat{y})$, $a_2 = a(\hat{x} - \hat{y})$, and $a_3 = \frac{1}{2}(a\hat{x} + a\hat{y} + c\hat{z})$. All spin directions are in the CuO $(x-y)$ plane. The $\xi$ axis is defined to be collinear with the spin directions.
FIG. 2. Nearest neighbor vectors connecting magnetic ions in a CuO plane. CuI spins are filled circles and CuII spins are open circles. Left: the vectors $\delta_x$ and $\delta_y$ between nearest neighboring CuI spins. Center: the vectors $\delta_x$ and $\delta_y$ which give the displacements of nearest neighboring CuI’s relative to a CuII. Right: the vectors $2\delta_x$ and $2\delta_y$ between nearest neighboring CuII spins.

FIG. 3. Spin-wave spectrum for wave vector $= q_x c/(2 \pi)$ along the $c$ direction in the absence of anisotropy. Each mode is two-fold degenerate. The left-hand scale applies to the lower modes and the right-hand scale applies to the optical mode. Left: without spin-wave interactions. In this case one mode has zero energy for arbitrary wave vector in the $c$ direction. Right: with spin-wave interactions. In the presence of easy plane anisotropy, the two-fold degeneracy is removed and only one mode (corresponding to rotation within the easy plane) is gapless at zero wave vector. When the four-fold in-plane anisotropy is also included there are no gapless modes.
FIG. 4. Interplanar CuII-CuII interactions. Left: a plaquette of CuII spins in one plane with a CuII neighbor in the adjacent plane over the center of the plaquette such that the isotropic CuII-CuII interaction is frustrated. Right: The principal axes for the exchange tensor of a spin in the $e$ sublattice at O with a spin in the $e$ sublattice at A. The directions of the axes are given in Eq. (80). The axes for the interactions of the spin at A with other spins in the lower plane can be obtained by a rotation of coordinates.
FIG. 5. Interplanar CuI-CuII interactions. Left: a plaquette of CuI spins in one plane with a CuII neighbor in the adjacent plane below the center of the plaquette such that the isotropic CuI-CuII interaction is frustrated. Right: The principal axes for the exchange tensor of a spin in the $e$ sublattice at O with a spin in the $c$ sublattice at A. The directions of the axes are given in Eq. (80). The axes for the interactions of other pairs of CuI-CuII nearest neighbors in adjacent planes can be obtained by a rotation of coordinates.

FIG. 6. Unit cell of the square lattice.
FIG. 7. Brillouin Zone for the Square Lattice.
TABLES

TABLE I. Definitions of Parameters. Notation: $\delta J \equiv (J_\perp - J_\parallel)/2$.

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$C_\alpha \approx 0.1686$</th>
<th>$\tau$</th>
<th>$\zeta$</th>
<th>$\xi$</th>
<th>$C_2 \approx 0.01^a$</th>
<th>$J_{12}^{(n)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$C_\alpha J^2_{12}/J$</td>
<td>Eq. (48)</td>
<td>$\delta J_1^2/J$</td>
<td>$(\delta J_{12})^2/J_2$</td>
<td>$(\delta J_2)^2/J_2$</td>
<td>Eq. (C19)</td>
<td>Eqs. (43), (46)</td>
</tr>
<tr>
<td>$\Delta K$</td>
<td>$\Delta K_{\text{eff}}$</td>
<td>$X = 7 \times 10^{-4} \text{Å}^{-3}$</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Eq. (83)</td>
<td>Eq. (87)</td>
<td>Eq. (85)</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

a) See Ref. 13.
TABLE II. Estimated Values of Parameters from Experiment and Theory

<table>
<thead>
<tr>
<th>Parameter</th>
<th>From Experiment</th>
<th>From Theory</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Value</td>
<td>Reference(^a)</td>
</tr>
<tr>
<td>(J)</td>
<td>130 ± 5</td>
<td>7</td>
</tr>
<tr>
<td>(J_3)</td>
<td>0.14 ± 0.02</td>
<td>5, 7</td>
</tr>
<tr>
<td>(J_{12})</td>
<td>-10 ± 2</td>
<td>7, TW</td>
</tr>
<tr>
<td>(J_2)</td>
<td>10.5 ± 0.5</td>
<td>5</td>
</tr>
<tr>
<td>(\Delta J_1(T = 0\text{K}))</td>
<td>0.081 ± 0.01</td>
<td>TW</td>
</tr>
<tr>
<td>(\Delta J_1(T = 200\text{K}))</td>
<td>0.068 ± 0.011</td>
<td>7</td>
</tr>
<tr>
<td>(\Delta J_{12})</td>
<td></td>
<td>(-1.3^d)</td>
</tr>
<tr>
<td>(\Delta J_2)</td>
<td>0.008 ± 0.004</td>
<td>16, TW</td>
</tr>
<tr>
<td>(\delta J_1)</td>
<td>±0.04</td>
<td>16, TW</td>
</tr>
<tr>
<td>(\delta J_{12})</td>
<td>±0.027</td>
<td>4</td>
</tr>
<tr>
<td>(\delta J_2)</td>
<td></td>
<td>0.4</td>
</tr>
<tr>
<td>(\Delta K_{\text{eff}})</td>
<td>TW</td>
<td>2.42 \times 10^{-4}</td>
</tr>
<tr>
<td>(\alpha)</td>
<td>0.135</td>
<td>5, 7, TW</td>
</tr>
<tr>
<td>(\tau)</td>
<td>(1.2 \times 10^{-5})</td>
<td>16, TW</td>
</tr>
<tr>
<td>(\zeta)</td>
<td>(7 \times 10^{-5})</td>
<td>4</td>
</tr>
<tr>
<td>(\xi)</td>
<td></td>
<td>10^{-6}</td>
</tr>
</tbody>
</table>

\(^a\) TW denotes this work.

\(^b\) This is the contribution to \(\delta J_{12}\) from dipolar interactions, which is much larger than that estimated from \(\delta J/J \sim 1.5 \times 10^{-4}\).

\(^c\) Evaluation based on the relevant \(J\)'s.

\(^d\) Evaluated for the similar compound Ba\(_2\)Cu\(_3\)O\(_4\)Cl\(_2\).
TABLE III. Renormalizations $J \rightarrow ZJ$

<table>
<thead>
<tr>
<th>Quantity</th>
<th>$J$</th>
<th>$J_3^a$</th>
<th>$\sqrt{J\Delta J_I}$</th>
<th>$\Delta K_{\text{eff}}^{(b)}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Renormalized to</td>
<td>$Z_cJ$</td>
<td>$\tilde{Z}_3J_3$</td>
<td>$Z_g\sqrt{J\Delta J_I}$</td>
<td>$\Delta K_{\text{eff}}^{(b)}\tilde{Z}_3^2$</td>
</tr>
<tr>
<td>$(1 + 0.085/S)J$</td>
<td>$(1 - 0.2/S)J_3$</td>
<td>$(1 - 0.2/S)\sqrt{J\Delta J_I}$</td>
<td>$(1 - 0.2/S)K_{\text{eff}}$</td>
<td></td>
</tr>
<tr>
<td>Refer to:</td>
<td>Ref. 20</td>
<td>Eq. (40)</td>
<td>Ref. 24</td>
<td>Eq. (99)</td>
</tr>
</tbody>
</table>

a) In the dynamics $JJ_3 \rightarrow Z_c\tilde{Z}_3JJ_3 \equiv Z_3^2JJ_3$.

b) In the dynamics $J_2\Delta K_{\text{eff}} \rightarrow Z_c\tilde{Z}_3J_2\Delta K_{\text{eff}} \equiv Z_3^2J_2\Delta K_{\text{eff}}$.

TABLE IV. Experimental Values of Spin-wave Gaps at Zero Wave Vector

<table>
<thead>
<tr>
<th>Mode</th>
<th>Temperature</th>
<th>Energy (meV)</th>
<th>Ref</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega_+$</td>
<td>$T = 200K$</td>
<td>5.5(3)$^a$</td>
<td>5</td>
</tr>
<tr>
<td>$\omega_-$</td>
<td>$T = 200K$</td>
<td>0.066(4)</td>
<td>16</td>
</tr>
<tr>
<td>$\omega_0^-$</td>
<td>$T \rightarrow 0K$</td>
<td>10.8(6)</td>
<td>7</td>
</tr>
<tr>
<td>$\omega_0$</td>
<td>$T \rightarrow 0K$</td>
<td>9.1(3)</td>
<td>7</td>
</tr>
<tr>
<td>$\omega_0^+$</td>
<td>$T \rightarrow 0K$</td>
<td>1.7473(4)</td>
<td>16</td>
</tr>
<tr>
<td>$\omega_0^-$</td>
<td>$T \rightarrow 0K$</td>
<td>1.72(20)</td>
<td>7</td>
</tr>
<tr>
<td>$\omega_0^+$</td>
<td>$T \rightarrow 0K$</td>
<td>0.149(3)</td>
<td>16</td>
</tr>
</tbody>
</table>

a) Extrapolated to $T = 0$. 

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TABLE V. Values in (10^{-6} \text{meV}) of the four-fold anisotropy constant \( k \)

<table>
<thead>
<tr>
<th>( k )</th>
<th>( T = 1.4 \text{K} )</th>
<th>( T = 100 \text{K} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>Experimental: From statics(^4)</td>
<td>25</td>
<td>2</td>
</tr>
<tr>
<td>Experimental: From dynamics(^5|^7)</td>
<td>41</td>
<td>1</td>
</tr>
<tr>
<td>Theoretical: See Eq. (108)</td>
<td>45</td>
<td>1(^a)</td>
</tr>
</tbody>
</table>

\(^a\) |\( J_\perp - J_\parallel \)| = 0.041 meV is fixed so that the dynamics and theory agree.

TABLE VI. Amplitude of the Dynamic Structure Factor

Results for wave vector \( 2\pi(H\hat{x}/a + K\hat{y}/a + L\hat{z}/c) + q_z \) for \( H \) and \( K \) half integral and \( H + K + L \) an even integer. Results are given only to leading order in \( J \). \( y_3 = x_3 + 4\Delta J_1 S + \frac{1}{2}\alpha = 2J_3 S[1 - \cos(q_z c/2)] + 4\Delta J_1 S + \frac{1}{2}\alpha \). The mode energies (without \( 1/S \) corrections) and intensities [\( I^{\alpha\beta}(\mathbf{q}) \)] are independent of the particular values of \( H, K, \) and \( L \) and are evaluated for \( q_z = 0 \).

<table>
<thead>
<tr>
<th>Mode Energy</th>
<th>Energy(^a) (meV)</th>
<th>Intensity</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \omega^&lt;_+ ) = ( [8JSy_3]^{1/2} )</td>
<td>10.8</td>
<td>( I^{zz}<em>{&gt;+} = 0 ), ( I^{nn}</em>{&gt;+} = 0 )</td>
</tr>
<tr>
<td>( \omega^&lt;_- ) = ( \left{ 8J_2 S \left[ 4\Delta J_2 S + \frac{2\alpha(4\Delta J_1 S + x_3)}{y_3} \right] \right}^{1/2} )</td>
<td>1.75</td>
<td>( I^{zz}<em>{&lt;+} = \frac{16J_2 S}{\omega^&lt;</em>+} ), ( I^{nn}_{&lt;+} = 0 )</td>
</tr>
<tr>
<td>( \omega^&gt;_+ ) = ( [8JS(x_3 + 2\alpha)]^{1/2} )</td>
<td>9.1</td>
<td>( I^{zz}<em>{&gt;\pm} = 0 ), ( I^{nn}</em>{&gt;\pm} = 0 )</td>
</tr>
<tr>
<td>( \omega^&lt;_- ) = ( \left[ 8J_2 S \frac{2\alpha x_3 + 64(2\pi - \xi)c_2 + 8\alpha S}{x_3 + \alpha} \right]^{1/2} )</td>
<td>0.15</td>
<td>( I^{zz}<em>{&lt;\pm} = 0 ), ( I^{nn}</em>{&lt;\pm} = \frac{16J_2 S}{\omega^&lt;_-} )</td>
</tr>
</tbody>
</table>

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a) See Table IV.

**TABLE VII. Amplitude of the Dynamic Structure Factor**

Results for wave vector $2\pi(H\hat{x}/a + K\hat{y}/a + L\hat{z}/c) + q_z$ for $H$ and $K$ integers and $H + K + L$ an even integer. The notation is as in Table V. Results are given only to leading order in $J$.

The intensities are evaluated for $q_z = 0$ and $H = L = 1$ and $K = 0$.

<table>
<thead>
<tr>
<th>Mode Energya (meV)</th>
<th>Formula for Intensity</th>
<th>Intensity</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\omega^\pi_1 = 10.8$</td>
<td>$I^{zz}_{&gt;+} = \frac{8JS}{\omega^\pi_1} \left[ 1 - (-1)^L \right]^2$</td>
<td>$50$</td>
</tr>
<tr>
<td>$\omega^\pi_1 = 10.8$</td>
<td>$I^{nn}_{&gt;+} = \frac{8JS}{\omega^\pi_1} \left[ \alpha [1 + (-1)^H] + (-1)^H \alpha \right]^2$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\omega^\pi_2 = 1.75$</td>
<td>$I^{zz}_{&lt;+} = \left[ 1 - (-1)^L \right]^2 \frac{(8JS)^2 (4JS) \alpha^2}{\omega^\pi_2 \omega^\pi_1}$</td>
<td>$26$</td>
</tr>
<tr>
<td>$\omega^\pi_2 = 1.75$</td>
<td>$I^{nn}_{&lt;+} = \frac{\omega^\pi_2}{4JS}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\omega^\pi_3 = 9.1$</td>
<td>$I^{zz}_{&gt;-} = \frac{8JS}{\omega^\pi_2} \left{ [1 - (1)^L] \left[ x_3 + \frac{1}{2} \alpha \right] + \alpha (1)^H \right}^2$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\omega^\pi_3 = 9.1$</td>
<td>$I^{nn}_{&gt;-} = \frac{8JS}{\omega^\pi_2} \left[ 1 - (-1)^L \right]^2$</td>
<td>$59$</td>
</tr>
<tr>
<td>$\omega^\pi_4 = 0.15$ meV</td>
<td>$I^{zz}_{&lt;+} = \frac{\omega^\pi_2}{4JS}$</td>
<td>$0$</td>
</tr>
<tr>
<td>$\omega^\pi_4 = 0.15$ meV</td>
<td>$I^{nn}_{&lt;+} = \frac{(4JS)(8JS)^2 \alpha^2}{\omega^\pi_2 \omega^\pi_3} \left[ 1 - (-1)^L \right]^2$</td>
<td>$570$</td>
</tr>
</tbody>
</table>

a) See Table IV.