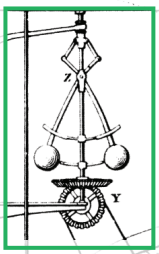
Physical Models of Living Systems

Probability, Simulation, Dynamics

Second edition



Philip Nelson

University of Pennsylvania
with the assistance of Sarina Bromberg,
Ann Hermundstad, Keith Kroma-Wiley,
and Jason Prentice

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Web resources

The book's Web site (`www.physics.upenn.edu/biophys/PMLS2e/`) contains links to the following resources in the Student section:

- *Datasets* contains datasets that are used in the problems. In the text, these are cited like this: Dataset 1, with numbers keyed to the list on the Web site.
- *Media* gives links to external media (graphics, audio, and video). In the text, these are cited like this: Media 1, with numbers keyed to the list on the Web site.
- *Errata* will appear as needed.

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