Topological Superconductors, Majorana Fermions and Topological Quantum Computation

1. Bogoliubov de Gennes Theory
2. Majorana bound states, Kitaev model
3. Topological superconductor
4. Periodic Table of topological insulators and superconductors
5. Topological quantum computation
6. Proximity effect devices

BCS Theory of Superconductivity

mean field theory : $\Psi^\dagger\Psi\Psi^\dagger\Psi \quad \Rightarrow \quad \langle \Psi^\dagger\Psi^\dagger \rangle \Psi\Psi = \Delta^*\Psi\Psi$

$$H = \frac{1}{2}\sum_{\mathbf{k}} \begin{pmatrix} \Psi^\dagger & \Psi \end{pmatrix} H_{BdG} \begin{pmatrix} \Psi \\ \Psi^\dagger \end{pmatrix}$$

Bogoliubov de Gennes Hamiltonian

$$H_{BdG} = \begin{pmatrix} H_0 & \Delta \\ \Delta^* & -H_0 \end{pmatrix}$$

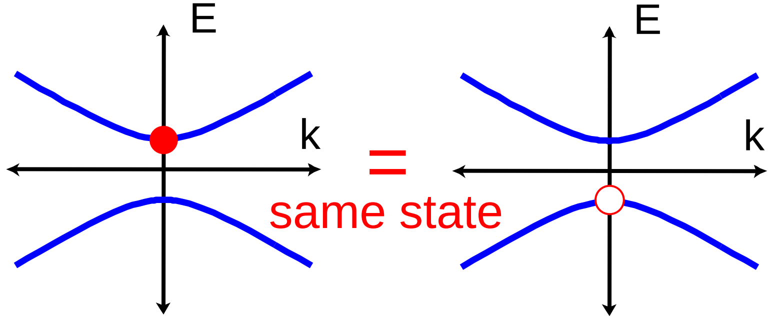
Intrinsic anti-unitary particle – hole symmetry

$$\Xi H_{BdG} \Xi^{-1} = -H_{BdG} \qquad \Xi\varphi = \tau_x \varphi^* \qquad \tau_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

$$\Xi^2 = +1$$

Particle – hole redundancy

$$\varphi_{-E} = \Xi\varphi_E \quad \Rightarrow \quad \gamma_E^\dagger = \gamma_{-E}$$



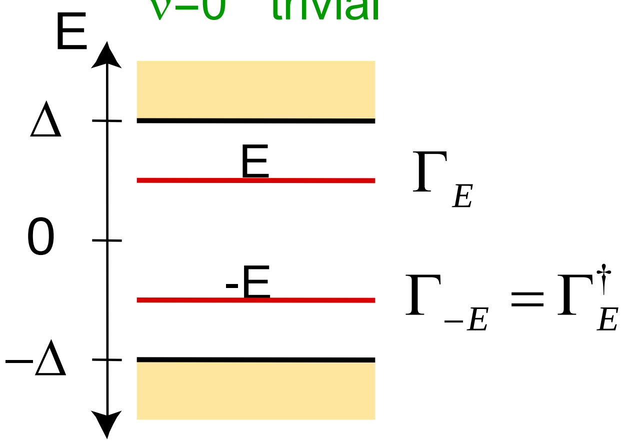
Bloch - BdG Hamiltonians satisfy $\Xi H_{BdG}(\mathbf{k})\Xi^{-1} = -H_{BdG}(-\mathbf{k})$

Topological classification problem similar to time reversal symmetry

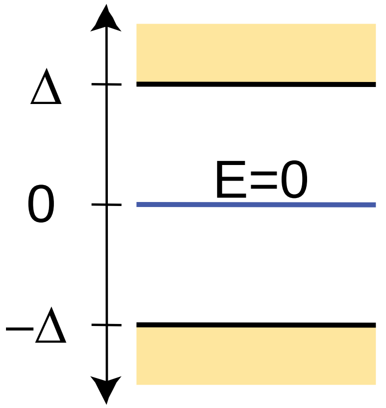
1D $\mathbb{Z}_2$ Topological Superconductor : $\nu = 0,1$ (Kitaev, 2000)

Bulk-Boundary correspondence : Discrete end state spectrum

$\nu=0$ "trivial"



$\nu=1$ "topological"



Zero mode

$$\Gamma_{E=0}^\dagger = \Gamma_{E=0} \equiv \gamma$$

Majorana fermion bound state

Majorana Fermion : Particle = Antiparticle $\gamma = \gamma^\dagger$

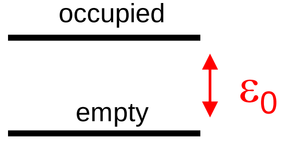
Real part of a Dirac fermion :
$$\begin{cases} \gamma_1 = \Psi + \Psi^\dagger & ; \quad \Psi = \gamma_1 + i\gamma_2 \\ \gamma_2 = -i(\Psi - \Psi^\dagger) & ; \quad \Psi^\dagger = \gamma_1 - i\gamma_2 \end{cases}$$

$$\gamma_i^2 = 1 \qquad \{\gamma_i, \gamma_j\} = 2\delta_{ij}$$

"Half a state"

Two Majorana fermions define a single two level system

$$H = 2i\varepsilon_0\gamma_1\gamma_2 = \varepsilon_0\Psi^\dagger\Psi$$



Mean Field: $$H = \frac{1}{2}\sum \begin{pmatrix} \Psi^\dagger & \Psi \end{pmatrix} H_{BdG} \begin{pmatrix} \Psi \\ \Psi^\dagger \end{pmatrix} \qquad H_{BdG} = \begin{pmatrix} H_0 & \Delta \\ \Delta^* & -H_0 \end{pmatrix}$$

Particle-hole symmetric spectrum of 1 body Hamiltonian H_{BdG}

$$H_{BdG}\left|\chi_n^\pm\right\rangle = \pm E_n \left|\chi_n^\pm\right\rangle \qquad \left|\chi_n^+\right\rangle = \Xi\left|\chi_n^-\right\rangle \qquad \left|\chi_n^+\right\rangle = \begin{pmatrix} u_n(\mathbf{r}) \\ v_n(\mathbf{r}) \end{pmatrix} \qquad \left|\chi_n^-\right\rangle = \begin{pmatrix} v_n^*(\mathbf{r}) \\ u_n^*(\mathbf{r}) \end{pmatrix}$$

Many body operators :

$$\Gamma_{n+} = \Gamma_{n-}^\dagger = \int d\mathbf{r}\left(u_n(\mathbf{r})\Psi(\mathbf{r}) + v_n(\mathbf{r})\Psi^\dagger(\mathbf{r})\right)$$

$$H = \frac{1}{2}\sum_n \begin{pmatrix} \Gamma_n{}^\dagger & \Gamma_n \end{pmatrix} \begin{pmatrix} E_n & 0 \\ 0 & -E_n \end{pmatrix} \begin{pmatrix} \Gamma_n \\ \Gamma_n{}^\dagger \end{pmatrix} = \sum_n E_n \Gamma_n{}^\dagger \Gamma_n$$

Zero Mode: $\left|\chi_0\right\rangle = \Xi\left|\chi_0\right\rangle \qquad u_0 = v_0^* \qquad \gamma_0 = \int d\mathbf{r}\left(u_0(\mathbf{r})\Psi(\mathbf{r}) + u_0^*(\mathbf{r})\Psi^\dagger(\mathbf{r})\right) = \gamma_0^\dagger$

Two zero modes : $\left|\chi_0^\pm\right\rangle = \left|\chi_0^+\right\rangle \pm i\left|\chi_0^-\right\rangle \qquad E_0^\pm = \pm\varepsilon$

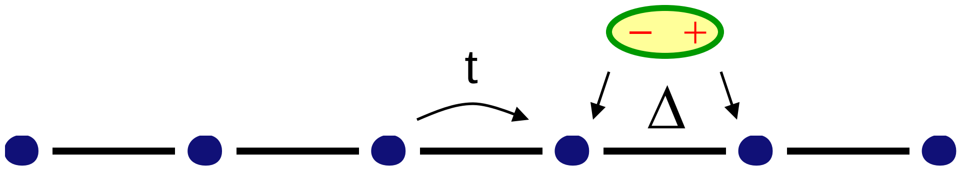
Bogoliubov QP operator : $\Gamma_{0+} = \Gamma_{0-}^\dagger = \gamma_L + i\gamma_R$

2 Majorana bound states = 1 Dirac fermion bound state = 1 qubit

Kitaev Model for 1D p wave superconductor

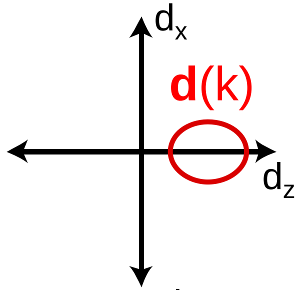
$$H-\mu N=\sum_i t(c_i^\dagger c_{i+1}+c_{i+1}^\dagger c_i)-\mu c_i^\dagger c_i+\Delta(c_i c_{i+1}+c_{i+1}^\dagger c_i^\dagger)$$

$$=\sum_k \begin{pmatrix}c_k^\dagger & c_{-k}\end{pmatrix} H_{BdG}(k)\begin{pmatrix}c_k\\ c_{-k}^\dagger\end{pmatrix}$$

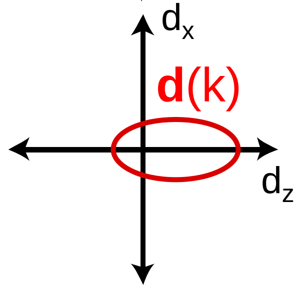


$$H_{BdG}(k)=\tau_z(2t\cos k-\mu)+\tau_x\Delta\sin k=\mathbf{d}(k)\cdot\vec{\tau}$$

$|\mu|>2t$: Strong pairing phase
trivial superconductor



$|\mu|<2t$: Weak pairing phase
topological superconductor



Similar to SSH model, except different symmetry : $(d_x,d_y,d_z)\big|_k=(-d_x,-d_y,d_z)\big|_{-k}$

Majorana Chain

$$c_i \to \gamma_{1i} + i\gamma_{2i}$$

$$\mu c_i^\dagger c_i \to 2i\mu\gamma_{1i}\gamma_{2i}$$

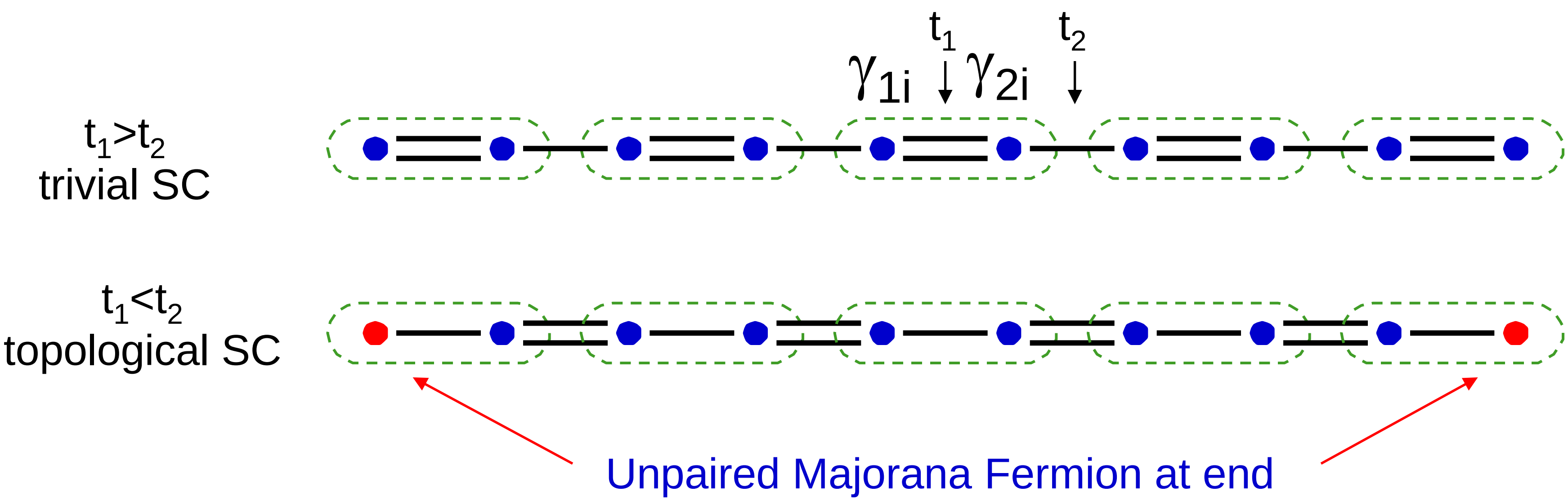
$$t\left(c_i^\dagger c_{i+1} + c_{i+1}^\dagger c_i\right) \to 2it\left(\gamma_{1i}\gamma_{2i+1} - \gamma_{2i}\gamma_{1i+1}\right)$$

$$\Delta\left(c_i c_{i+1} + c_{i+1}^\dagger c_i^\dagger\right) \to 2i\Delta\left(\gamma_{1i}\gamma_{2i+1} + \gamma_{2i}\gamma_{1i+1}\right)$$

$$H = 2i\sum_i t_1\gamma_{1i}\gamma_{2i} + t_2\gamma_{2i}\gamma_{1i+1}$$

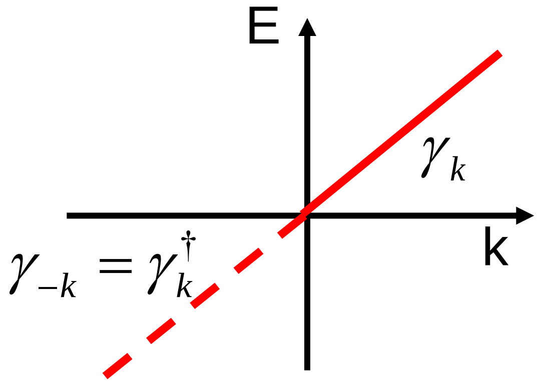
For Δ=t : nearest neighbor Majorana chain

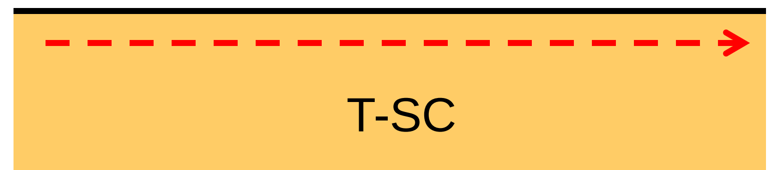
$$t_1 = \mu,\ t_2 = 2t$$



2D $\mathbb{Z}$ topological superconductor (broken T symmetry)

Bulk-Boundary correspondence: n = # Chiral Majorana Fermion edge states





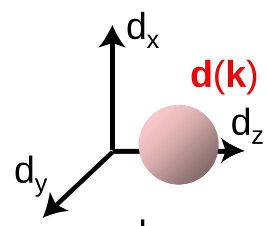
Examples

- Spinless p_x+ip_y superconductor (n=1)
- Chiral triplet p wave superconductor (eg Sr_2RuO_4) (n=2)

Read Green model : $H = \sum_{\mathbf{k}} \left(\frac{\mathbf{k}^2}{2m} - \mu \right) c_{\mathbf{k}}^\dagger c_{\mathbf{k}} + \left(\Delta(\mathbf{k}) c_{\mathbf{k}} c_{-\mathbf{k}} + c.c. \right)$ $\quad \Delta(\mathbf{k}) = \Delta_0 \left(k_x + i k_y \right)$

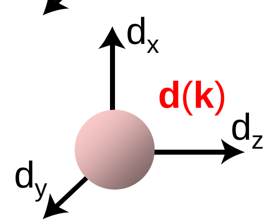
Lattice BdG model : $H_{BdG}(\mathbf{k}) = \tau_z \left(2t \left[\cos k_x + \cos k_y \right] - \mu \right) + \Delta \left(\tau_x \sin k_x + \tau_y \sin k_y \right) = \mathbf{d}(k) \cdot \vec{\tau}$

|μ|>4t : Strong pairing phase
trivial superconductor



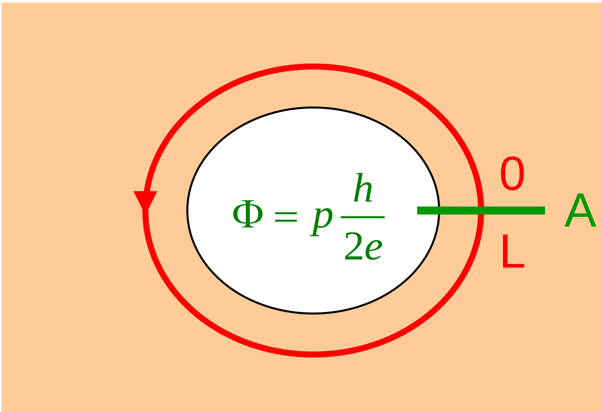
Chern number 0

|μ|<4t : Weak pairing phase
topological superconductor



Chern number 1

Majorana zero mode at a vortex

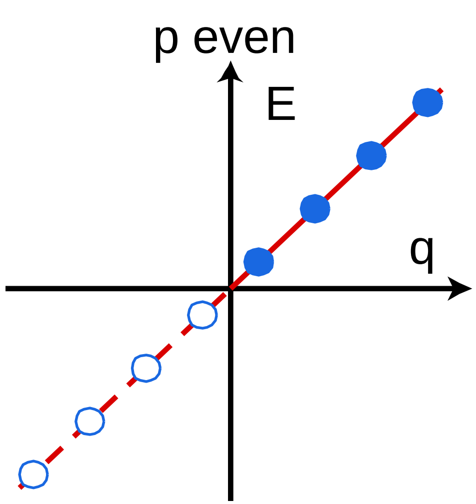


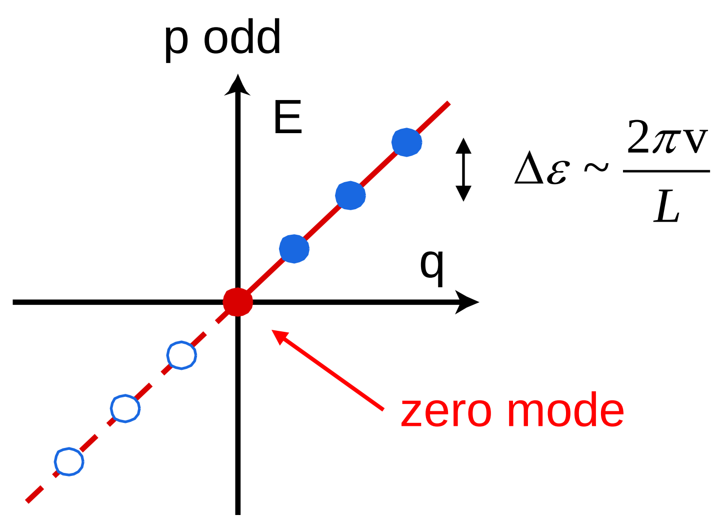
Hole in a topological superconductor threaded by flux

Boundary condition on fermion wavefunction

$$\psi(L) = (-1)^{p+1}\psi(0)$$

$$\psi(x) \propto e^{iq_m x} \quad ; \quad q_m = \frac{\pi}{L}\left(2m+1+p\right)$$





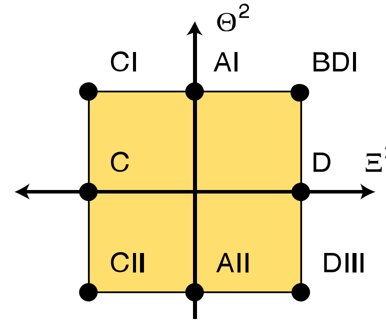
Without the hole : Caroli, de Gennes, Matricon theory ('64) $\quad \Delta\varepsilon \sim \dfrac{\Delta^2}{E_F}$

Periodic Table of Topological Insulators and Superconductors

Anti-Unitary Symmetries :

- Time Reversal : $\Theta H(\mathbf{k})\Theta^{-1} = +H(-\mathbf{k})$; $\Theta^2 = \pm 1$
- Particle - Hole : $\Xi H(\mathbf{k})\Xi^{-1} = -H(-\mathbf{k})$; $\Xi^2 = \pm 1$

Unitary (chiral) symmetry : $\Pi H(\mathbf{k})\Pi^{-1} = -H(\mathbf{k})$; $\Pi \propto \Theta\Xi$



8 antiunitary symmetry classes

Symmetry				d							
AZ	Θ	Ξ	Π	1	2	3	4	5	6	7	8
A	0	0	0	0	$\boxed{\mathbb{Z}}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$
AIII	0	0	1	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0	$\mathbb{Z}$	0
AI	1	0	0	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$
BDI	1	1	1	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$
D	0	1	0	$\boxed{\mathbb{Z}_2}$	$\boxed{\mathbb{Z}}$	0	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$
DIII	-1	1	1	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$	0
AII	-1	0	0	0	$\boxed{\mathbb{Z}_2}$	$\boxed{\mathbb{Z}_2}$	$\mathbb{Z}$	0	0	0	$\mathbb{Z}$
CII	-1	-1	1	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0	0
C	0	-1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0	0
CI	1	-1	1	0	0	$\mathbb{Z}$	0	$\mathbb{Z}_2$	$\mathbb{Z}_2$	$\mathbb{Z}$	0

Complex K-theory

Real K-theory

Altland-Zirnbauer Random Matrix Classes

Bott Periodicity d→d+8

Kitaev, 2008
Schnyder, Ryu, Furusaki, Ludwig 2008

Majorana Fermions and Topological Quantum Computing

(Kitaev '03)

The degenerate states associated with Majorana zero modes define a topologically protected quantum memory

- 2 Majorana separated bound states = 1 fermion $\Psi = \gamma_1 + i\gamma_2$
  - 2 degenerate states (full/empty) = 1 qubit
- 2N separated Majoranas = N qubits
- Quantum Information is stored non locally
  - Immune from local decoherence

Braiding performs unitary operations

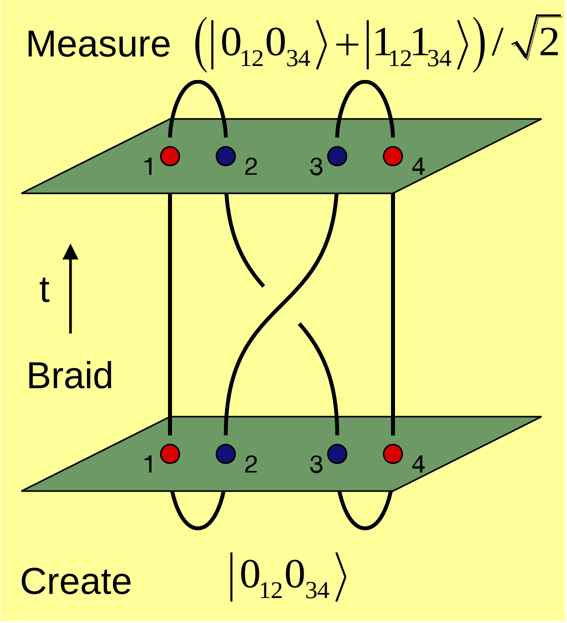
Non-Abelian statistics

Interchange rule (Ivanov 03)

$$\gamma_i \to \gamma_j$$

$$\gamma_j \to -\gamma_i$$

These operations, however, are not sufficient to make a universal quantum computer



Potential condensed matter hosts for Majorana bound states

- Quasiparticles in fractional Quantum Hall effect at ν=5/2 Moore Read '91
- Unconventional superconductors

  - Sr_2RuO_4 Das Sarma, Nayak, Tewari '06
  - Fermionic atoms near feshbach resonance Gurarie '05

- Proximity Effect Devices using ordinary s wave superconductors

  - Topological Insulator devices Fu, Kane '08
  - Semiconductor/Magnet devices Sau, Lutchyn, Tewari, Das Sarma '09, Lee '09, ...

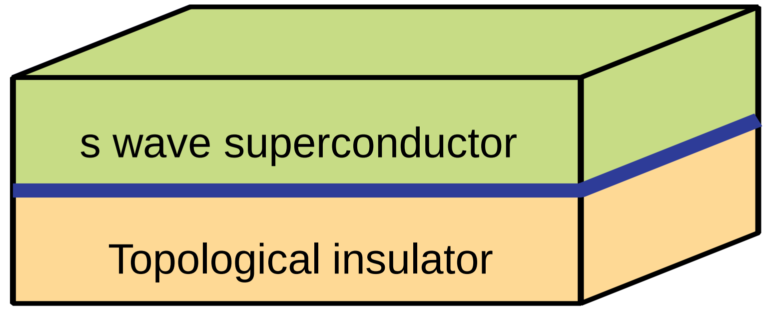
- .... among others

Current Status : Not Observed

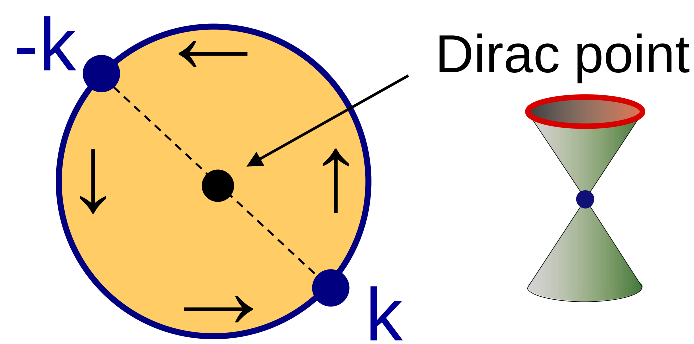
Superconducting Proximity Effect

$$H = \psi^{\dagger}(-iv\vec{\sigma}\cdot\vec{\nabla} - \mu)\psi$$
$$+\Delta_S \psi_{\uparrow}^{\dagger}\psi_{\downarrow}^{\dagger} + \Delta_S^{*}\psi_{\downarrow}\psi_{\uparrow}$$

proximity induced superconductivity
at surface

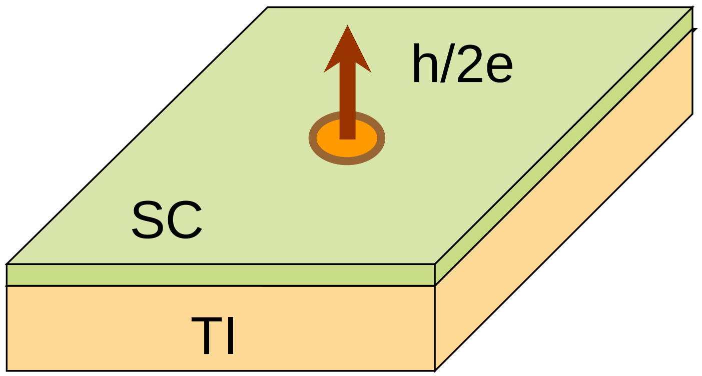


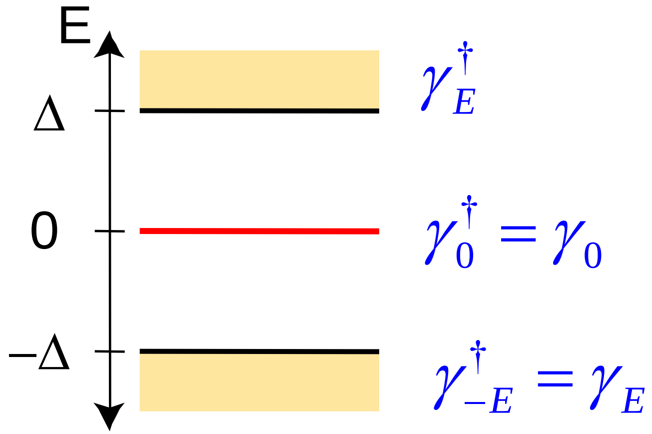
- Half an ordinary superconductor
- Similar to spinless p_x+ip_y superconductor, except :
  - Does not violate time reversal symmetry
  - s-wave singlet superconductivity
  - Required minus sign is provided by π Berry's phase due to Dirac Point
- Nontrivial ground state supports Majorana fermion bound states at vortices



Majorana Bound States on Topological Insulators

1. h/2e vortex in 2D superconducting state

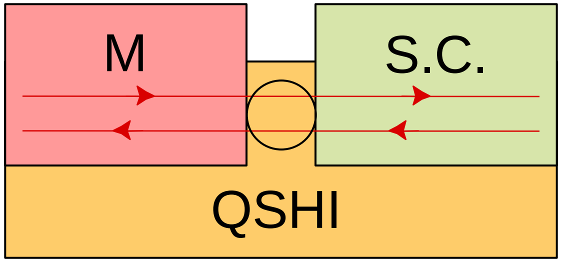




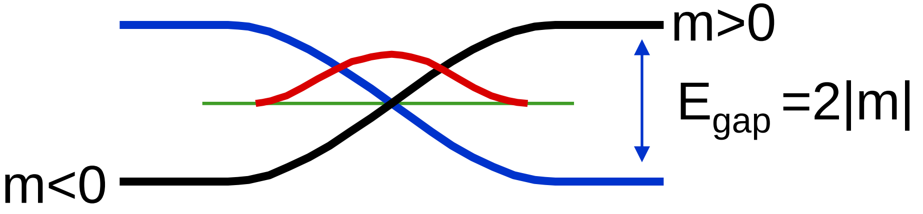
Quasiparticle Bound state at E=0

Majorana Fermion γ_0 "Half a State"

2. Superconductor-magnet interface at edge of 2D QSHI



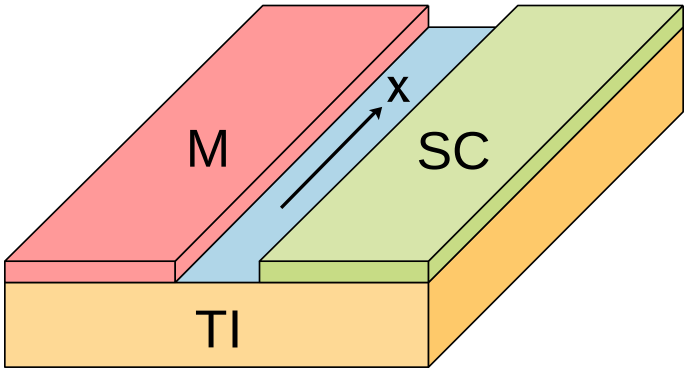
$$m = |\Delta_S| - |\Delta_M|$$

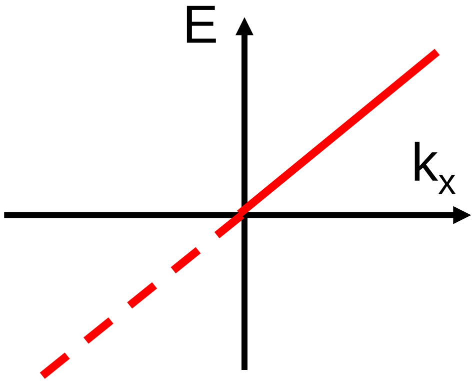


Domain wall bound state γ_0

1D Majorana Fermions on Topological Insulators

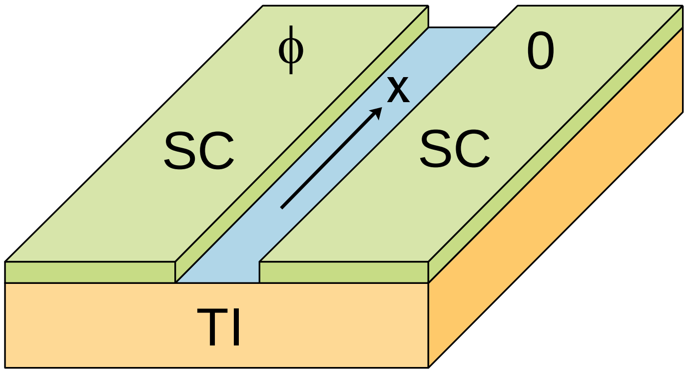
1. 1D Chiral Majorana mode at superconductor-magnet interface

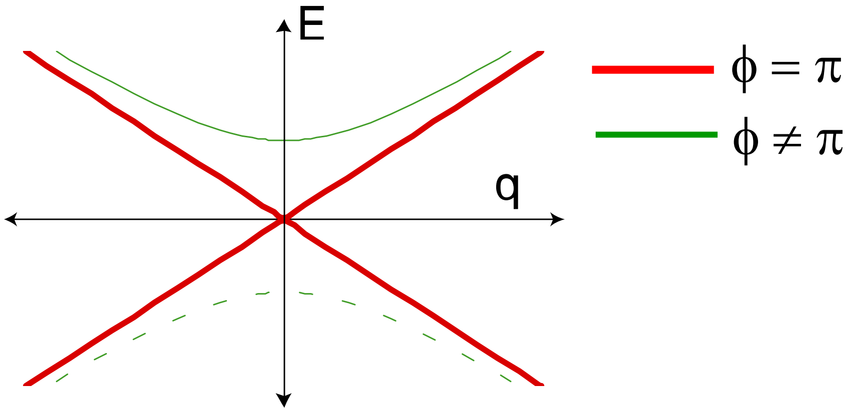




$\gamma_k = \gamma^\dagger_{-k}$: "Half" a 1D chiral Dirac fermion $\quad H = -i\hbar v_F \; \gamma \partial_x \gamma$

2. S-TI-S Josephson Junction





Gapless non-chiral Majorana fermion for phase difference $\phi = \pi$

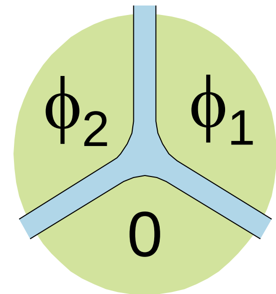
$$H = -i\hbar v_F \left(\gamma_L \partial_x \gamma_L - \gamma_R \partial_x \gamma_R\right) + i\Delta \cos(\phi/2)\gamma_L \gamma_R$$

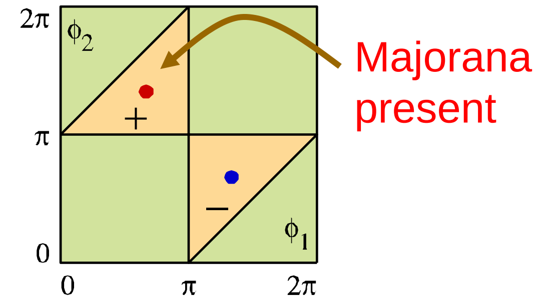
Manipulation of Majorana Fermions

Control phases of S-TI-S Junctions

Tri-Junction :

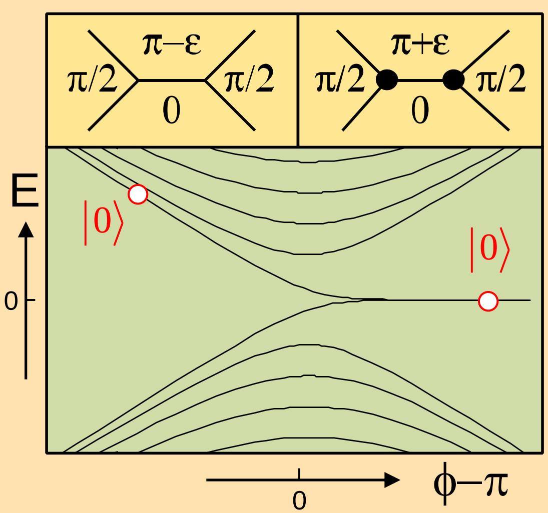
A storage register for Majoranas





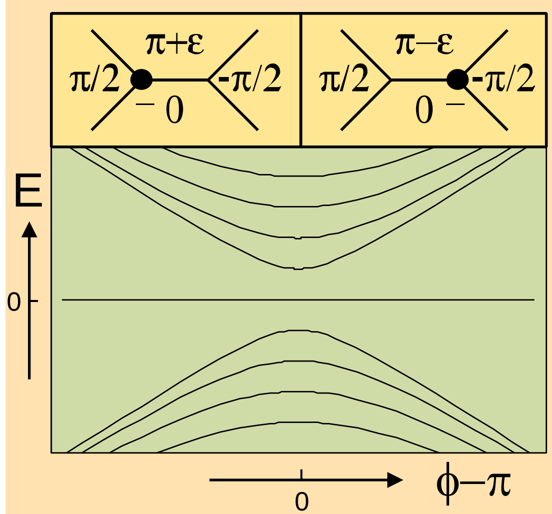
Create

A pair of Majorana bound states can be created from the vacuum in a well defined state |0>.



Braid

A single Majorana can be moved between junctions. Allows braiding of multiple Majoranas

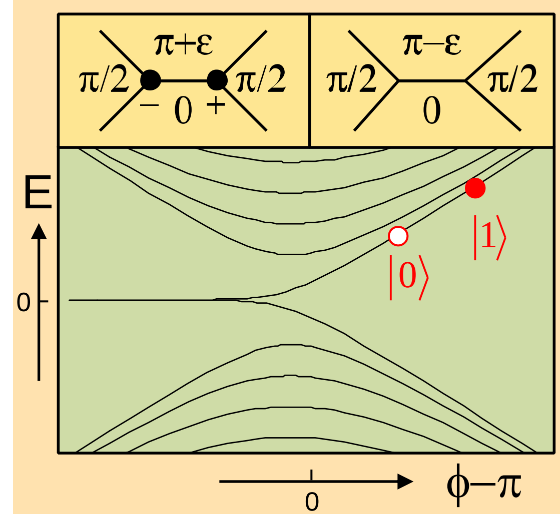


Measure

Fuse a pair of Majoranas.

States |0,1> distinguished by

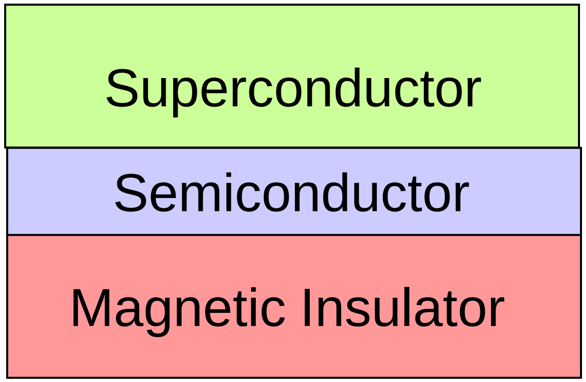
- presence of quasiparticle.
- supercurrent across line junction

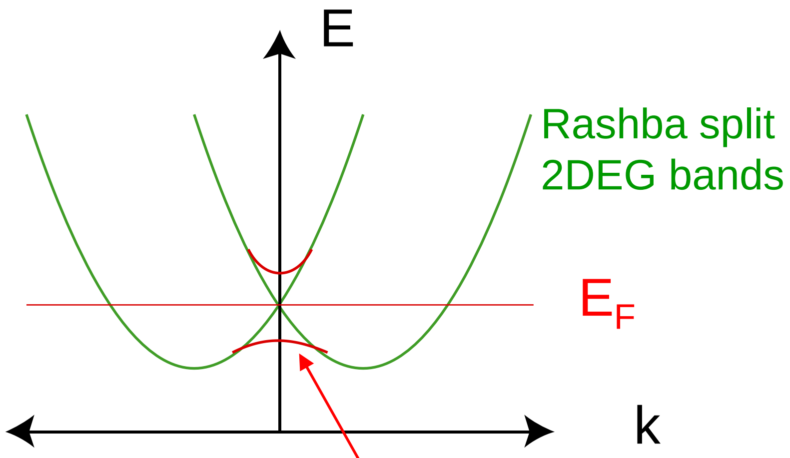


Another route to the 2D p+ip superconductor

Semiconductor - Magnet - Superconductor structure

Sau, Lutchyn, Tewari, Das Sarma '09





- Single Fermi circle with Berry phase $\pi-\varepsilon$
- Topological superconductor with Majorana edge states and Majorana bound states at vortices.
- Variants :
  - use applied magnetic field to lift Kramers degeneracy (Alicea '10)
  - Use 1D quantum wire (eg InAs). A route to 1D p wave superconductor with Majorana end states. (Oreg, von Oppen, Alicea, Fisher '10
- Challenge : requres very low electron density $\rightarrow$ high purity.