General validity of Jastrow-Laughlin wave functions

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We construct a class of interacting-boson Hamiltonians whose exact ground-state wave functions are of Jastrow form. These Hamiltonians generally have both two- and three-body interactions; however, we show that the three-body interaction does not affect the long-wavelength physics. This enables us to deduce that (a) for Coulomb interacting bosons at $T=0$ the lower critical dimension is $d_c=2$; (b) in two dimensions, the ground-state wave function has the form of the modulus of the Laughlin wave function and exhibits algebraic long-range order; and (c) for short-range repulsions, we obtain a simple expression for the sound velocity. We also show that the Laughlin wave function is the generic ground-state wave function for fermions in a magnetic field corresponding to a filling factor of $\nu=1/q$.

A fruitful approach to some many-body problems is to write down a wave function, which has much of the anticipated physics built in, and ask the following question: For what kind of Hamiltonian $\mathcal{H}_0$ is this wave function the exact ground state? Hopefully, $\mathcal{H}_0$ can be shown to be perturbatively connected to the Hamiltonian $\mathcal{H}$ which one sets out to solve. In particular, if one can show that by writing $\mathcal{H}=\mathcal{H}_0+V$, the "perturbation" V does not change the long-distance physics, then one can say that V is an "irrelevant" perturbation and therefore $\mathcal{H}_0$ faithfully represents the dynamics of the problem in the regime of physical interest. In this sense, the proposed ground state solves the problem.

In this paper, we apply this type of reasoning to interacting-boson systems. In particular, we will show that Jastrow[1] wave functions, which have been extremely successful in describing the long-wavelength physics of Bose systems,[2] are indeed the exact ground-state wave functions of a class of Hamiltonians that are perturbatively related to the Bose Hamiltonians involving kinetic energy and pairwise repulsion between the particles. As a result, the Jastrow wave function gives correlation functions which have the same asymptotic behavior as the ones computed from the exact wave functions. Therefore, as far as the long-distance physics is concerned, Jastrow wave functions are generic ground-state wave functions of interacting boson Hamiltonians. We go on to apply the same logic to the problem of pairwise short-range interacting spinless fermions moving in two space dimensions under an external magnetic field. We show that, in exactly the same sense, the Laughlin wave function[3] is the generic ground-state wave function of that problem for filling factor $\nu=1/q$ (q an odd integer).

We begin by asking the question: Given a many boson wave function $\psi_B(\{r_i\})$, for what kind of interaction will ψ_B be the exact ground state? Given a Hamiltonian of the form

$$\mathcal{H}_0=-\frac{1}{2m}\sum_i \nabla_i^2+U(\{r_i\}) , \tag{1}$$

if we choose $U(\{\mathbf{r}_i\})=(1/2m)\sum_i \psi_B^{-1}\nabla_i^2\psi_B$, then ψ_B will be an eigenstate. Furthermore, if we specify that ψ_B is positive definite, then we are guaranteed that ψ_B is the ground state of $\mathcal{H}_0$. For a general boson wave function, $U(\{\mathbf{r}_i\})$ is a very complicated many-particle interaction. The form of $U(\{\mathbf{r}_i\})$ is tremendously simplified when ψ_B has the Jastrow form,

$$\psi_B(\{\mathbf{r}_i\})=\exp\left[-\sum_{i<j} f(\mathbf{r}_{ij})\right] . \tag{2}$$

In that case $U(\{\mathbf{r}_i\})$ is a combination of two- and three-body interactions,

$$U=\frac{1}{2m}\sum_i \left[\sum_{j\neq i,\ k\neq i} \nabla_i f(r_{ij})\cdot\nabla_i f(r_{ik})-\sum_{j\neq i}\nabla_i^2 f(r_{ij})\right] . \tag{3}$$

The interacting boson problems that we are interested in do not generically involve a three-body interaction. Sutherland[4] has used this type of logic to show that in one dimension, for a particular case of Jastrow wave functions with $f(r)=\gamma\ln r$, the three-body interaction vanishes, so that the Jastrow wave function is the exact ground state of the problem with a two-body potential $V(r)=\gamma^2/2mr^2$. In this paper, we go further and argue that, in general, even if there is a three-body interaction, $\mathcal{H}_0$ has the same long-wavelength behavior as a Hamiltonian involving only pairwise interaction. For this purpose it is convenient to rewrite $U(\{r_i\})$ in a second quantized form:

$$\begin{aligned}U=&-\frac{1}{2m}\int d^dr_1 d^dr_2 \nabla_1^2 f(r_{12})\rho(\mathbf{r}_1)\rho(\mathbf{r}_2)\\&+\frac{1}{2m}\int d^dr_1 d^2r_2 d^dr_3 \nabla_1 f(r_{12})\cdot\nabla_1\\&\times f(r_{13})\rho(\mathbf{r}_1)\rho(\mathbf{r}_2)\rho(\mathbf{r}_3) ,\end{aligned}$$

where $\rho(\mathbf{r})\equiv\psi^\dagger(\mathbf{r})\psi(\mathbf{r})$ is the density operator. To proceed, we write $\rho(\mathbf{r})\equiv\bar{\rho}+\delta\rho(\mathbf{r})$, where $\bar{\rho}$ is the mean particle density. By direct substitution it is simple to show that

$$\mathcal{H}_0 = -\frac{1}{2m}\int d^d r\, \psi^\dagger(\mathbf{r})\nabla^2\psi(\mathbf{r}) + \tfrac{1}{2}\int d^d r_1 d^d r_2 V_2(r_{12})\delta\rho(\mathbf{r}_1)\delta\rho(\mathbf{r}_2) + \int d^d r_1 d^d r_2 d^d r_3 V_3(\mathbf{r}_{12},\mathbf{r}_{13}) \times \delta\rho(\mathbf{r}_1)\delta\rho(\mathbf{r}_2)\delta\rho(\mathbf{r}_3)\ , \quad (4)$$

where

$$V_2(r_{12}) \equiv \int d^d r_3 \frac{\bar{\rho}}{m}\nabla_3 f(r_{31})\cdot\nabla_3 f(r_{32}) - \frac{1}{m}\nabla_1^2 f(r_{12})\ ,$$
$$V_3(\mathbf{r}_{12},\mathbf{r}_{13}) \equiv \frac{1}{2m}\nabla_1 f(r_{12})\cdot\nabla_1 f(r_{13})\ . \quad (5)$$

Equations (4) and (5) constitute one of the central results of this paper: we have constructed a Hamiltonian whose exact ground-state wave function has Jastrow form.

Let us for the moment drop the three-body term in (5) and analyze the long-wavelength properties of the resulting two-body Hamiltonian $\mathcal{H}_2$. It is well known that for Bose fluid at zero temperature the long-wavelength elementary excitations are exhausted by a single collective mode.[5] Therefore as far as the long-wavelength physics is concerned, it is a good approximation to restrict ourselves to the sub-Hilbert space spanned by this collective mode and study the effects of the three-body perturbation—the single-mode approximation (SMA).

A convenient representation for doing this is the coherent-state path-integral formulation of the problem, in which we write the partition function as

$$Z_2 = \int D(\bar{\psi}\psi)\exp[-S_2(\bar{\psi},\psi)]\ ,$$

with

$$S_2 = \int_0^\beta d\tau \int d^d r\, \bar{\psi}(\mathbf{r})\left[\partial_\tau - \frac{\nabla^2}{2m}\right]\psi(\mathbf{r}) + \tfrac{1}{2}\int d^d r_1 d^d r_2 V_2(r_{12})[\rho(\mathbf{r}_1)-\bar{\rho}][\rho(\mathbf{r}_2)-\bar{\rho}]\ . \quad (6)$$

In this language, the SMA amounts to treating longitudinal fluctuations in the phase of ψ as the only low-energy dynamical degrees of freedom. Specifically, we write $\psi = \rho^{1/2}e^{i\theta}$ and ignore vortex configurations in the phase θ. We then integrate out the massive $\delta\rho(\mathbf{r},\tau) = \rho(\mathbf{r},\tau) - \bar{\rho}$ fluctuations and keep only the leading order $\nabla\theta$ terms. At the saddle point, $\delta\rho(\mathbf{q},\omega_\nu) = i\omega_\nu\theta(\mathbf{q},\omega_\nu)/V_2(q)$. Upon Fourier transformation we obtain the single-mode effective action

$$S_2 = \frac{1}{2}\frac{1}{\beta}\sum_{\omega_\nu}\int \frac{d^d q}{(2\pi)^d}\left[\frac{\omega_\nu^2}{V_2(q)} + \frac{\bar{\rho}q^2}{m}\right] \times \theta(\mathbf{q},\omega_\nu)\theta(-\mathbf{q},\omega_\nu)\ , \quad (7)$$

where $\omega_\nu \equiv = 2\pi\nu/\beta$ is the Bose-Matsubara frequency and $V_2(q)$ is the spatial Fourier transform of $V_2(r)$. Upon Wick rotation $\omega_\nu \to i\omega$ we obtain the collective mode dispersion,

$$\omega_q = [V_2(q)\bar{\rho}q^2/m]^{1/2}\ .$$

By computing the equal-time density-density correlation function,

$$\langle \delta\rho(\mathbf{q})\delta\rho(-\mathbf{q})\rangle_{T=0} \equiv \int \frac{d\omega}{2\pi}\frac{\omega^2}{V_2(q)^2}\langle \theta(\mathbf{q},\omega)\theta(-\mathbf{q},-\omega)\rangle\ ,$$

we obtain the structure factor

$$S(q) \equiv \frac{1}{\bar{\rho}}\langle \delta\rho(\mathbf{q})\delta\rho(-\mathbf{q})\rangle = \{q^2/[4\bar{\rho}V_2(q)m]\}^{1/2}\ .$$

This is simply the familiar relation $\omega_q = q^2/[2mS(q)]$ obtained in the SMA.[3]

We now analyze the effects of V_3. Clearly, if there is a gap in the collective-mode spectrum, the density fluctuations will be massive, so that V_3 will not affect the physics for $\mathbf{q},\omega \to 0$. We are actually able to make a stronger statement. even if there is no gap for density fluctuations, we shall now show that V_3 is an irrelevant perturbation in the renormalization-group sense, so that it does not affect the long-wavelength physics of the model.

In terms of θ, the three-body interaction will make a contribution to the action at $T = 0$,

$$g\int \frac{d^d q_1 d\omega_1}{(2\pi)^{d+1}}\frac{d^d q_2 d\omega_2}{(2\pi)^{d+1}}\frac{V_3(q_1,q_2)\omega_1\omega_2\omega_3}{V_2(q_1)V_2(q_2)V_2(q_3)} \times \theta(\mathbf{q}_1,\omega_1)\theta(\mathbf{q}_2,\omega_2)\theta(\mathbf{q}_3,\omega_3)\ , \quad (8)$$

where $q_3 \equiv -q_1 - q_2$ and $\omega_3 = -\omega_1 - \omega_2$. In order to address the relevance of V_3 upon scaling, we have inserted a coupling constant g. Our argument rests on an analysis of the canoncial dimension of g.[6] If $f(q) \to q^{-\chi}$ $(\chi > 0)$ at small q, then both V_2 and $V_3 \to q^{2-2\chi}$. If we let $q' = bq$ and $\omega' = b^z\omega$ ($b > 1$ is a scaling factor), then from Eq. (7) we deduce that the dynamical exponent $z = 2 - \chi$. Therefore $\theta' = b^{-(d+2+z)/2}\theta$ and hence $g' = b^{-(d+\chi)/2}g$. Under the renormalization group, the strength of the three-body interaction will decrease as we lower the energy and momentum cutoff. It is in this sense that the long-wavelength physics of the problem is entirely captured by (6) because the importance of V_3 diminishes as we go to lower and lower q and ω.

Some disclaimers are in order here. When we say that "the long-wavelength physics is determined by the two-body Hamiltonian," we mean that if we compute the long-distance and/or long-time correlation functions and the $q \to 0$ collective mode dispersion via SMA using the Jastrow wave functions, we should obtain identical asymptotic behavior as we would have obtained using the exact ground-state wave functions. This simply means that if $\langle \psi^\dagger(\mathbf{r},\tau)\psi(\mathbf{r}',\tau)\rangle \to |r - r'|^{-\eta}$ and $\omega_q \to q^\epsilon$, our procedure enables us to obtain the exponents η and ϵ correctly. Since we have no handle on the short-distance behavior and hence have no way to obtain the exact value for quantities that are nonuniversal, we are not solving the problem exactly. Moreover, when the repulsion is sufficiently strong and/or long range, so that, for instance, the bosons will form a charge-density wave, this perturbative analysis, and in particular the approximation of ignoring vortices, may no longer be valid.

Now let us imagine we are given an interacting boson problem with a two-body potential $V_2(r)$, and we want to construct a ground-state wave function that reproduces the correct $q \to 0$ physics. What we do is to write down a Jastrow wave function given by (1) with $f(r)$ determined via (5), which upon Fourier transform gives

$$V_2(q) = \frac{\bar{\rho}}{2m} q^2 f(q) f(-q) + \frac{1}{2m} q^2 f(q)$$

or

$$f(q) = \frac{1}{2\bar{\rho}} [-1 + (1 + 8\bar{\rho} m V_2(q)/q^2)^{1/2}] .$$

Suppose $V_2(q) \to V q^{-\gamma}$ for small q and $\gamma > -2$. Then it follows that for small q, $f(q) \to [2mV_2(q)/(q^2\bar{\rho})]^{1/2}$, which in the SMA gives a structure factor $S(q) = 1/f(q)$.

We first consider the case $\gamma = 0$, which corresponds to a short-range interaction. In this case, $f(q) \to 1/q$ so that the Jastrow function $f(r)$ decays with distance as $r^{-(d-1)}$ in d dimensions.[7] The SMA predicts a gapless phonon mode $\omega_q = \sqrt{V\bar{\rho}/m}\,|q|$. At low densities, this sound velocity agrees with the well-known result;[8] however, at higher densities the three-body interaction will introduce perturbative corrections.

For Coulomb interacting bosons, $\gamma = 2$, and we have $f(q) \to 1/q^2$. There is a gap in the collective-mode corresponding to a plasma frequency $\omega_p = \sqrt{V\bar{\rho}/m}$. For the three-dimensional Coulomb interaction, $V = 4\pi e^2$, this gives the familiar result. In two dimensions, the Jastrow function is

$$f(r) = \frac{1}{2\pi} \left[\frac{2mV}{\bar{\rho}} \right]^{1/2} \ln|r| .$$

In addition, for Coulomb interactions, it is necessary to include a one-body potential, which serves as a neutralizing background for V. When this is done, we conclude that for a boson Hamiltonian

$$\mathcal{H} = -\frac{1}{2m} \sum_i \nabla_1^2 + \frac{\pi\bar{\rho}}{m} Q^2 \sum_{i \neq j} \ln(r_{ij}) + V_{\text{background}} ,$$

the Jastrow wave function

$$\phi(\{z_i\}) = \prod_{i \neq j} |z_i - z_j|^Q \exp\left[\frac{1}{4 l_0^2} \sum_j |z_j|^2 \right]$$

produces the correct asymptotic behaviors in the long-distance and long-time correlation. For a circular geometry, $V_{\text{background}} = (\pi^2 \bar{\rho}^2 Q^2/m) \sum_i r_i^2$, and the effective "magnetic length" $l_0^2 = (2\pi\bar{\rho}Q)^{-1}$. This wave function is precisely the modulus of the Laughlin wave function for the fractional quantum-Hall effect at filling factor $\nu = 1/Q$. Given this wave function Girvin and McDonald[9] showed that the single-particle reduced density matrix $\rho_1(z-z')$ exhibits algebraic off-diagonal long-range order so long as Q is sufficiently small that the ground state does not Wigner crystallize:[10]

$$\rho_1(z-z') \to |z-z'|^{-Q/2} .$$

We thus conclude that in two dimensions, logarithmically interacting bosons exhibit algebraic long-range order. In three space dimensions $f(r) \to 1/r$, the same analysis implies that for boson interacting via repulsive $1/r$ Coulomb interaction, the single-particle off-diagonal density matrix decays as $\exp(-1/r)$, which implies the existence of off-diagonal long-range order. Similarly, in one dimension, the same quantity decays as $\exp(-r)$, so that there is no off-diagonal long-range order.

We now turn to the problem of short-range interacting spinless fermions moving in two space dimensions in an external magnetic field. For field strength that corresponds to a filling factor $\nu = 1/m$ (m an odd integer), the fermions "condense" into an incompressible quantum liquid state, which gives rise to the phenomenology of the fractional quantum-Hall effect.[11] In this state the longitudinal conductivity vanishes in the dc limit and the transverse conductivity assumes the quantized value e^2/hm. A wave function that contains all the essential physics was proposed by Laughlin,[3]

$$\psi_L^m(\{z_i\}) = \prod_{i \neq j} (z_i - z_j)^m \exp\left[\sum_k |z_k|^2 \right] , \tag{9}$$

where $z_j \equiv x_j + iy_j$ is the dimensionless complex coordinate of the particle.

Our purpose in this section is to show that the Laughlin wave function is a generic form describing the incompressible quantum liquid of spinless fermions at filling factor $\nu = 1/m$ in the same sense that the Jastrow wave function is the generic form describing the superfluid of spinless bosons. To proceed, we note that it is possible to translate a fermionic problem into a problem of hard-core bosons by performing the singular gauge transformation,[8]

$$\psi_F \equiv \prod_{i<j} \frac{(z_i - z_j)^k}{|z_i - z_j|^k} \phi_B(\{z_i\}) . \tag{10}$$

Here k is an odd integer, which guarantees the antisymmetry of the total wave function. ϕ_B, on the other hand, is a symmetric Bose wave function. In simple terms (9) means that in two space dimension we can view a spinless fermion as a hard-core boson carrying k quanta of fictitious, "statistical" flux.

Girvin and MacDonald[9] were the first to realize that although the reduced density matrix $\rho_1(z-z)$ computed from (8) decays as a Gaussian with distance, the corresponding quantity calculated from ϕ_B (with $k = m$) exhibits algebraic long-range order. This long-range order follows from the fact that the bosons see no net magnetic field. On the average, the statistical flux cancels the external magnetic field.[12] This cancellation is not exact, however, since the statistical flux is tied to the particles. This will introduce interactions between the bosons, since the bosons feel a fluctuating magnetic field which is determined by the fluctuations in the boson density.[13]

We may obtain the effective interacting boson Hamiltonian by substituting (10) into the Schrödinger equation for ψ_F,

$$\left[-\sum_j \frac{\nabla_j^2}{2m} + \frac{1}{2} \sum_{i \neq j} U(z_i - z_j) \right] \psi_F = E_0 \psi_F .$$

We obtain an effective Hamiltonian for ϕ_B, which in second quantized form reads

$$\mathcal{H}_B=-\frac{1}{2m}\int d^2r\,\psi^\dagger[\nabla-i\,\mathbf{A}(\mathbf{r})+i\,\mathbf{A}_S(\mathbf{r})]^2\psi+\tfrac{1}{2}\int d^2r_1d^2r_2\,U(r_{12})\rho(\mathbf{r}_1)\rho(\mathbf{r}_2)\,, \tag{11}$$

where $\mathbf{A}_S(r)=k\int d^2r'\hat{\mathbf{z}}\times\nabla G(r-r')\rho(\mathbf{r}')$ is the statistical vector potential [$\hat{\mathbf{z}}$ is the unit vector perpendicular to the physical plane, $G(r)$ is the two-dimensional Green's function satisfying $\nabla^2G(r)=-2\pi\delta(r)$].

At a magnetic field corresponding to $\nu=1/k$, the external vector potential will be canceled (up to a gauge transformation) by $k\int d^2r'\hat{\mathbf{z}}\times\nabla G(r-r')\bar{\rho}$. Therefore, if we substitute $\rho(\mathbf{r})=\bar{\rho}+\delta\rho(\mathbf{r})$ into (9), we obtain

$$\mathcal{H}_B=-\frac{1}{2m}\int d^2r\,\psi^\dagger\Big[\nabla-ik\hat{\mathbf{z}}\times\int d^2r'\nabla G(r-r')\delta\rho(\mathbf{r}')\Big]^2\psi+\tfrac{1}{2}\int d^2r\,d^2r'\,U(r-r')\delta\rho(\mathbf{r})\delta\rho(\mathbf{r}')\,. \tag{12}$$

Straightforward manipulation gives

$$\mathcal{H}_B=-\frac{1}{2m}\int d^2r\,\psi^\dagger\nabla^2\psi+\tfrac{1}{2}\int d^2r_1d^2r_2\,V_2(r_{12})\delta\rho(\mathbf{r}_1)\delta\rho(\mathbf{r}_2)+\int d^2r_1d^2r_2\,\tilde{V}_2(r_{12})\delta\rho(\mathbf{r}_1)\hat{\mathbf{z}}\cdot\nabla\times\mathbf{j}(\mathbf{r}_2)+\int d^2r_1d^2r_2d^2r_3\,V_3(\mathbf{r}_{12},\mathbf{r}_{13})\delta\rho(\mathbf{r}_1)\delta\rho(\mathbf{r}_2)\delta\rho(\mathbf{r}_3)\,, \tag{13}$$

with $\mathbf{j}=1/2im\,[\psi^\dagger\nabla\psi-(\nabla\psi^\dagger)\psi]$ and

$$V_2(r_{12})=k^2\frac{\bar{\rho}}{m}G(r_{12})+U(r_{12})\,,$$
$$\tilde{V}_2(r_{12})=kG(r_{12})\,, \tag{14}$$
$$V_3(\mathbf{r}_{12},\mathbf{r}_{13})=\frac{k^2}{2m}\nabla_1G(r_{12})\cdot\nabla_1G(r_{13})\,.$$

We now proceed to analyze the long-wavelength behavior of (11) in the same manner as above in the framework of the SMA. In this case, since we ignore vortices and set $\mathbf{j}(\mathbf{r})=\nabla\theta(\mathbf{r})/m$, the term involving $\tilde{V}_2$ has no effect. Provided that the interaction $U(\mathbf{r})$ is short range, the most important part of $V_2(\mathbf{r})$ at long distances is $k^2(\bar{\rho}/m)G(r_{12})=2\pi k^2(\bar{\rho}/m)\ln r_{12}$. If we apply the same logic as that following Eq. (8) we conclude that V_3 is irrelevant to the long-distance physics. The ground-state wave function ψ_B is therefore well described by a Jastrow function with $f(r)=k\ln r$. This gives us precisely

$$\phi_B=|\psi_L^k|\,.$$

If we now undo the singular gauge transformation (10), we see that $\psi_F=\psi_L^k$. Thus we see that when cast in the form of an interacting boson problem, the Laughlin wave function is precisely the Jastrow wave function for Coulomb interacting bosons and is in the same sense the generic wave function for the odd denominator fractional quantum-Hall state. Of course, when the range of the interaction between particles is increased, Haldane[14] has shown that the Laughlin wave function ceases to be the ground state. This signifies the failure of our approximation of ignoring vortices.

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