Edge state transport

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The rich low energy structure at the edge of fractional quantum Hall fluids provides an ideal arena for the study of strongly correlated low dimensional systems. The chiral Luttinger liquid model offers a general and powerful framework for describing these edge excitations. Here we review recent theoretical progress, focusing on edge state transport. The chiral Luttinger model is first introduced for the integer quantum Hall state at filling $\nu = 1$ and the Laughlin states at filling $\nu = 1/m$ (m odd), where there is only a single edge mode. For the fractional quantum Hall effect this edge mode is a strongly correlated Luttinger liquid, which leads to a number of striking predictions for the transport behavior through a point contact. For $\nu = 1/3$ the point contact conductance is predicted to vanish as T^4 at low temperatures, whereas for resonant tunneling a universal lineshape with width scaling to zero as $T^{2/3}$ is predicted. A recent experiment by Milliken, Umbach and Webb gives evidence for this striking behavior. For hierarchical quantum Hall fluids, the edge is more complicated, consisting of multiple modes. In this case impurity scattering at the edge is argued to play an essential role, allowing the different modes to equilibrate with one another. In the absence of such equilibration the Hall conductance can be non universal. We show that edge impurity scattering leads to a new disorder dominated phase. The low energy physics in this phase is described by a new, exactly soluble, fixed point. Various experimental predictions and implications which follow from this exact solution are described in detail.

PACS numbers: 72.10.-d 73.20.Dx

I. INTRODUCTION

The most striking feature of the quantum Hall effect is the remarkably precise quantization of the Hall conductance[1]. It was in 1980 that von Klitzing[2] observed unanticipated plateaus in the Hall conductance of a two-dimensional electron gas, precisely quantized at integer multiples of the fundamental unit, e^2/h. This spectacular result was so surprising that a nobel prize was awarded to von Klitzing for his discovery of the integer quantum Hall effect (IQHE) within five years of the discovery. Only two years later, in 1982, the fractional quantum Hall effect was discovered[3].

Already in 1981 Laughlin proposed an appealing and general explanation of the precise quantization in the IQHE[4]. Laughlin's argument, as elaborated on by Halperin[5], focussed on an annulus which was threaded by a time dependent magnetic flux - a Corbino disk geometry. Each time the flux was increased by one flux quantum, an electron was argued to be adiabatically transferred from the inside to the outside edge of the annulus. This resulted in a flow of electrical current proportional to the electro-motive driving force, with a precisely quantized coefficient of proportionality - the Hall conductance. While theoretically appealing the connection with experimental transport measurements was somewhat unclear. In particular, the experimental samples were not multi-connected, but rather had a single outer edge, to which multiple current and voltage probes were attached. The important current carrying edge states, elucidated by Halperin[5], could then carry the transport current between adjacent contacts along the edge. In contrast to the Corbino-disk geometry, a bulk current was, at least in principle, unnecessary.

Laughlin's wave function[6] for the FQHE was the critical step in identifying the origin of plateaus with fractional Hall conductance. At filling $\nu = 1/m$, for odd integer m, the wave function described a featureless fluid with a gap to all excitations. The lowest lying excitations were shown to be fractionally charged quasiparticles, with charge νe. Within a Corbino-disk geometry, the quantized Hall conductance could be understood as a discrete transfer of such quasiparticles between the inner and outer edges, one for each magnetic flux quantum threading the bore of the disk. But again, there were questions of relevance to experimental geometries. Since there was an energy gap for quasiparticle creation, should one really expect the current to be carried by them? Also, as in the IQHE, there were nagging questions about the role of disorder. Impurity scattering was believed to be important for giving the Hall plateaus an observable width, but was also expected to destroy the incompressibility, and lead to low-lying, possibly localized, bulk excitations. These, however, had better not destroy the precise quantization!

The difficulties in relating the microscopic Laughlin wave function to the measured quantized conductance, were reminiscent of the situation in superconductivity soon after BCS theory. While leading to many verifiable microscopic predictions, the connection of the BCS wave function to the macroscopic behavior of superconductors - the Meissner effect and zero resistance even with impurity scattering - was not altogether apparent. In fact, the macroscopic behavior of superconductors follows more directly from effective theories, such as Ginzburg-Landau theory. Guided by this a number of theorists sought to develop a more phenomenological approach to the FQHE. The key step was undertaken by Girvin and MacDonald[7] who emphasized a striking analogy between Laughlin's wave function and superfluidity. As earlier emphasized by Halperin[8], in Laughlin's $\nu = 1/3$ wave function three vortices, or zeroes in the wave function, sit on each electron. Such binding of electrons to vortices was believed to be a universal feature of the incompressible Hall fluid. Girvin and MacDonald pointed out that the electron/vortex composite, when viewed as a new "particle", is not a Fermion, but rather has bosonic statistics. Moreover, these composite boson particles were shown to be bose condensed in Laughlins state, exhibiting (algebraic) off-diagonal long-ranged order, much as in a conventional superfluid. This appealing picture was placed on a firm theoretical foundation by Zhang Hanson and Kivelson[9] and Read[10], who developed a full-blown Ginzburg-Landau like description of the quantum Hall fluid. The superfluidity accounted naturally for the dissipationless flow in the QHE, and the bound vortices dragged along by the current generated a quantized Hall voltage. However, these effective theories did not initially address the question of realistic finite geometries, and so the relation with the quantized conductance measured in transport experiments remained unclear.

During this period of intense interest in effective theories of the FQHE, Buttiker was re-considering the role of edge states in the IQHE[11]. He emphasized that experiments with complicated geometries involving multiple current and voltage probes, could be easily interpreted in terms of transmission and reflection of edge states. In Landauer transport theory[12], generalized by Büttiker[13] to multiple contacts, the conductance is expressed in terms of the transmission of electron waves incident from the leads at the Fermi energy. Within this framework, for a fluid of non-interacting electrons within an IQHE state, all of the transport current is confined to flow along the edges of the sample. This approach offered a simple and unifying picture of numerous transport experiments in the IQHE. But being a theory for non-interacting electrons, it was unclear how to generalize the edge-state approach to the FQHE. Following early work by Chang[14], Beenakker[15] and MacDonald[16], a pioneering step was made by Wen, who developed a general theory for the edge excitations in the FQHE[17]. This theory rested firmly on the Ginzburg-Landau description of the bulk FQHE.

In this paper we offer an overview of the edge-state approach to transport in the FQHE. While providing a simple and direct understanding of the quantized Hall plateaus, the edge excitations of a fractional Hall fluid are extremely interesting in their own right. As emphasized by Wen[17], the FQHE edge excitations cannot be described in terms of non-interacting electrons. Rather, they must be thought of as a fluid of fractionally charged quasiparticle excitations. For the Laughlin sequence of fractions, $\nu = 1/m$, the edge can be viewed as a gas of Laughlin quasiparticles, which is liberated and free to move along the sample edge. Wen emphasized the close analogy between these edge excitations, and the low energy excitations in models of interacting one-dimensional (1d) electron gases.

It was over a quarter of a century earlier that pioneering theoretical work first revealed the profound effects that electron interactions can have on a 1d electron gas[18–20]. Even weak interactions were found to destablize a Fermi-liquid description of the 1d gas. Instead, the 1d gas exhibited a novel phase, termed a "Luttinger liquid" some years later by Haldane[21]. In a Luttinger liquid the low energy excitations are not weakly dressed electrons, but are collective density waves, moving to the right and left without scattering. Wen emphasized that the edge excitations in a FQHE state at $\nu = 1/m$, which move only in one direction (say right), are formally equivalent to the right moving half of a Luttinger liquid. He coined the term "chiral Luttinger liquid" to describe such edge excitations[17]. The Luttinger liquid has remained primarily a theoretical curiosity, since even with modern lithographic techniques it is very difficult to fabricate clean one-channel quantum wires. However, the current carrying FQHE edge states, which are unaffected by disorder (see below) provide a unique laboratory for the study of "ideal" Luttinger liquids.

For the Laughlin sequence $\nu = 1/m$, the FQHE edge is predicted to consist of a single branch chiral Luttinger liquid[17]. In this case, the IQHE edge-state transport theory can be readily generalized. The quantization of the conductance follows readily, even in the presence of disorder. In the hierarchical FQHE states, multiple edge excitations are predicted - in some cases with edge modes moving in both directions. For example, for $\nu = 2/3$ two oppositely moving edge modes are predicted[16,17,22]. This disagrees, however, with a recent time domain experiment[24] in which only a single propagating mode was observed. Moreover, in these cases, the edge state theory predicts a Hall conductance which is neither universal nor quantized[23]! The conductance depends on microscopic details, such as the strength of the electron interaction between the two edge modes. This is clearly an unsatisfactory state of affairs.

For such hierarchical states it is absolutely essential to incorporate impurity scattering at the edge. Edge impurity-scattering transfers charge between the channels, as depicted in Fig. 1. When the channels move in the same direction (Fig. 1a), interchannel charge transfer does not affect the net transmission. In contrast, charge transfer between oppositely moving edge channels (Fig. 1b), modifies the net transmitted edge current and conductance. As shown below, for $\nu = 2/3$ such impurity scattering drives an edge phase transition which separates a weakly and strongly disordered edge phase. An exact solution in the disorder dominated phase[23], reveals the existence of two modes: A charge mode which gives an appropriately quantized Hall conductance, $(2/3)e^2/h$, and a neutral mode, propagating in the opposite direction. Since an electron is a superposition of the charge and neutral modes, it should be possible to excite the neutral mode by tunneling an electron into the edge. With suitable time domain experiments it might thus be possible to directly detect the neutral mode. At finite temperatures the neutral mode is predicted to decay, with a rate varying as T^2, behavior reminiscent of a quasiparticle in a Fermi liquid. The exact solution for the disorder dominated phase of the $\nu = 2/3$ edge, can be readily generalized to a broad class of FQHE fluids[25], at filling $\nu = n/(np + 1)$, with integer n and even integer p.

Unfortunately, the rich low energy structure at the edge of FQHE fluids is not readily exposed via bulk transport measurements. Much more revealing are laterally confined samples, in which different edges of a given sample are brought into close contact, allowing for inter edge tunneling. The simplest case is that of a single point contact or constriction, in an otherwise bulk Hall fluid, as depicted in Fig. 2. In this case, backscattering between edge states on the top and bottom edges can occur via tunneling at the point contact. The magnitude of this backscattering can be inferred by simply measuring the conductance, or transmission, through the point contact.

For a $\nu = 1/m$ Hall fluid, in which each edge has only one mode, a point contact is in fact isomorphic to a point scatterer in a one-channel 1d electron gas. Moreover, for $\nu = 1$ the electron gas is a non-interacting Fermi liquid, whereas for fractional ν the gas is a Luttinger liquid. Point contacts in Hall fluids thus provide an ideal arena to study experimentally the differences between Fermi and Luttinger liquids. In particular, since the transport current through the point contact depends on the tunneling density of states (DOS) for each edge mode, by varying the temperature one can effectively use transport to directly extract the energy dependent DOS of a Luttinger liquid. In striking contrast to the IQHE where the conductance through the point contact should go to a constant at low temperatures, the conductance for a FQHE fluid with $\nu = 1/m$ is predicted[26,27] to vary strongly with temperature, vanishing algebraically at low temperatures, $G(T) \sim T^{(2/\nu)-2}$. This can be attributed to the suppressed tunneling density of states in a Luttinger liquid[28]. Preliminary evidence for this signature of a Luttinger liquid has been seen in recent experiments by Milliken, Umbach and Webb[29]. In the presence of two adjacent defects, one has the possibility for resonant tunneling through an isolated localized state. In contrast to resonances in the IQHE, which are expected to have temperature independent Lorentzian line-widths at low temperatures, for $\nu = 1/3$ the resonance widths are predicted[27,30] to vanish with temperature as $T^{2/3}$. Moreover, at low temperatures the resonance is predicted to be strongly non-Lorentzian with a completely universal lineshape. This lineshape has been computed by Monte Carlo simulations[27] and more recently by Bethe Ansatz methods[31], and is in reasonable agreement with measurements by Milliken et. al.[29].

The point contact geometry in Fig. 2 should also be useful as a probe of the composite edge structure of hierarchical Hall fluids. Indeed, the presence of the predicted neutral modes, at fillings such as $\nu = 2/3$, leads to a modification in the temperature dependence of the conductance through the point contact[23]. Although indirect, such an experiment could thus provide evidence of the exotic neutral edge excitations.

This paper is organized as follows. Section II is devoted to a brief overview of the chiral Luttinger-liquid edge state theory for clean IQHE and FQHE edges. In Section 3 we consider the effects of impurity scattering of hierarchical states which have multiple edge modes, focusing on the inter-mode equilibration. Transport through a point contact in the FQHE is discussed in Section 4, as a probe of Luttinger liquids. Section 5 is devoted to a brief summary and discussion, emphasizing experimental implications.

II. EDGE STATES

A. IQHE

Consider first a non-interacting electron gas moving in two-dimensions and confined by a potential $V(y)$, which is constant (say zero) for $|y| < W/2$, where W is the width of the system, and rises rapidly for larger values of $|y|$. A magnetic field of strength B is taken perpendicular to the $x - y$ plane. In the Landau gauge with vector potential $A_x = By$ the Hamiltonian takes the form

$$H = \frac{1}{2m}(p_x + \frac{e}{c}By)^2 + \frac{1}{2m}p_y^2 + V(y), \qquad (2.1)$$

with momentum $p_\mu = -i\hbar\partial_\mu$. Being separable eigenstates can be written

$$\psi(x, y) = \frac{1}{2\pi}e^{ikx}\Phi(y) \qquad (2.2)$$

where $\Phi(y)$ satisfies

$$[-\frac{\hbar^2}{2m}\partial_y^2 + \frac{1}{2}m\omega_c^2(y - y_0)^2 + V(y)]\Phi(y) = E\Phi. \qquad (2.3)$$

Here the cyclotron frequency is $\omega_c = eB/mc$, and $y_0 = k\ell^2$, with the magnetic length $\ell = \sqrt{\hbar/m\omega_c}$. If the confining potential is taken as slowly varying, $\partial_y V(y) << \hbar\omega_c/\ell$, then the potential $V(y)$ in (2.3) can be approximated by the constant $V(y_0)$, and the eigenfunctions are then simply harmonic oscillator wave functions, Φ_n centered at y_0. Thus wavefunctions which satisfy the time dependent Schrodinger equation take the form,

$$\psi(x,y,t) = \frac{1}{2\pi} e^{i(kx-\omega_k t)} \Phi_n(y - k_x \ell^2) \tag{2.4}$$

with energies

$$\hbar\omega_k = (n + \frac{1}{2})\hbar\omega_c + V(k\ell^2). \tag{2.5}$$

The eigenstates are plane waves in the x-direction with dispersion ω_k, sketched in Fig. 3.

When the Fermi energy, μ, lies between bulk Landau levels, the only low lying excitations are at the edges, where $\hbar\omega_k = \mu$. For one full bulk Landau level, there are then only two Fermi points, a right moving one which is confined to the top edge, and a left mover confined to the bottom edge. The low energy physics is isomorphic to a one-dimensional non-interacting electron gas, which likewise has two Fermi points, at $\pm k_F$. At low energies it is legitimate to linearize the spectrum so that the wave functions take the form $\psi \sim e^{ik(x-vt)}$ with velocity $v = \partial\omega_k/\partial k$.

The quantization of the Hall conductance in the IQHE can be very easily understood in terms of edge states, as emphasized by Büttiker[11]. Imagine raising the chemical potential (or Fermi energy) of the "source" electrode by an amount eV, while keeping the "drain" electrode at μ. The top edge state, being injected from the source electrode, will be at higher chemical potential and carry more current. The additional current can be expressed as

$$I = ev\delta n \tag{2.6}$$

where $\delta n = \delta k/2\pi$ is the change in the (1d) electron density on the top edge. Since the velocity $v = \delta\omega/\delta k$ this can be re-written as

$$I = \frac{e}{2\pi}\delta\omega_k = GV \tag{2.7}$$

with a conductance $G = e^2/h$. Notice that the conductance is independent of the velocity v, depending only on fundamental constants. More generally, for n-full Landau levels there will be n edge modes, each contributing a quantized value, e^2/h, to the conductance. The edge state approach to quantum transport can be suitably generalized to more complicated geometries, with multiple current and voltage probes[11].

An appealing feature of the edge theory of the quantized conductance is the insensitivity to disorder. Impurity scattering at the edge cannot degrade the source-to-drain current, since all of the n edge modes move in the same direction. The only effect of scattering is an unimportant forward scattering phase shift. Backscattering is only possible when the opposite edges are brought into close proximity, so that inter-edge tunneling becomes feasible. The quantized conductance is also insensitive to electron-electron interactions, as we will argue below. For some hierarchical FQHE states multiple edge modes which move in both directions on each edge are predicted. In these cases impurity scattering, rather than being unimportant, is in fact essential in order to explain the observed quantized conductance.

Before generalizing to FQHE edge states, it will be useful to discuss a second-quantized formulation of the IQHE edge. For simplicity, consider an edge with only one mode, corresponding to one full Landau level. The low energy states may be described by linearizing (2.5) about the Fermi wavevector to obtain,

$$H = v \int \frac{dk}{2\pi} k\psi^\dagger(k)\psi(k). \tag{2.8}$$

Here we have taken the Fermi energy to be the zero of energy and k_F to be the zero of momentum. It is also useful to transform to real space, which leads to

$$H = v \int dx \psi^\dagger(x) i\partial_x \psi(x). \tag{2.9}$$

where $\psi(x)$ is a (1d) Fermion field operator, which satisfies the usual anticommutation relations, $[\psi(x), \psi^\dagger(x')]_- = \delta(x - x')$. It will often be convenient to consider a Grassman path integral for the associated partition function. The appropriate Euclidean action is

$$S = \int dx d\tau \psi^*(\partial_\tau + iv\partial_x)\psi \tag{2.10}$$

where τ is imaginary time.

As discussed, each free fermion edge channel contributes a conductance of one, in units of e^2/h. However, in the FQHE the conductance is fractional, $G = \nu e^2/h$. For this reason a free fermion description of FQHE edge states is not possible. Rather, an appropriate description is in terms of a bosonic field, roughly analogous to the displacement field for phonons in a solid. Before discussing this, we first show how the IQHE edge can be "bosonized"[32–34]. While unnecessary for the IQHE edge, the bosonized description is useful since it can be easily generalized to describe FQHE edges.

By way of motivation, consider the nature of the low energy edge excitations of a quantum Hall fluid. The incompressible Hall fluid can support long wavelength fluctuations at its edge - analogous to water waves on the surface of the sea. However, unlike water waves, the edge waves can propagate only in one direction, which is determined by the sign of the magnetic field. Classically, this can be understood as an $E \times B$ drift of the electrons caused by the edge confining potential. A quantum mechanical description may be developed in terms of the 1d edge density operator, defined as

$$\rho(x) =: \psi^\dagger(x)\psi(x) : \tag{2.11}$$

where the dots denote a normal ordering with respect to the filled Fermi sea of the Hamiltonian (2.8). It is convenient to work with a new field ϕ defined by

$$\rho(x) = \frac{1}{2\pi}\partial_x \phi(x). \tag{2.12}$$

¿From the continuity equation for charge conservation the electron edge current is then

$$I = \frac{e}{2\pi}\dot{\phi} \tag{2.13}$$

By carefully accounting for the normal ordering with an appropriate "point splitting", one can show that the operator $\phi(x)$ obeys the Kac Moody commutation relation[32,34,33]

$$[\phi(x), \phi(x')] = i\pi \mathrm{sgn}(x - x'). \tag{2.14}$$

Notice that the momentum $\Pi(x)$ conjugate to $\phi(x)$ is not independent, but rather given by $\Pi(x) = (1/2\pi)\partial_x\phi$.

The electron operator at the edge may be constructed by noting that removing a charge is equivalent to creating a 2π instanton in the field configuration $\phi(x)$. This may be accomplished via

$$\psi(x) = e^{i2\pi \int^x dx' \Pi(x')} \tag{2.15}$$
$$= \quad e^{i\phi(x)}. \tag{2.16}$$

Using the Kac-Moody algebra, one can readily show that the fermion anticommutation relations are obeyed, $\psi(x)\psi(x') = -\psi(x')\psi(x)$.

After normal ordering the Hamiltonian (2.8) - subtracting an appropriate constant from the energy so that the filled Fermi sea has zero energy - it can be reexpressed in terms of the density as[32,34,33]:

$$H = \pi v \int dx \rho^2(x). \tag{2.17}$$

Upon using the conjugate momentum to obtain the Lagrangian, one arrives at the appropriate bosonized action, which in imaginary time is

$$S = \frac{1}{4\pi}\int dx d\tau \partial_x \phi (i\partial_\tau + v\partial_x)\phi. \tag{2.18}$$

The first term reflects the Kac-Moody commutation relations. Clearly, this action describes modes which propagate in one direction at a velocity v.

It is instructive to re-derive the quantized conductance for the IQHE using the bosonized action (2.18). To this end, consider an edge state which flows between two reservoirs which are in equilibrium at different chemical potentials (see Fig. 4). We model the reservoirs by considering an infinite edge, in which the "sample" resides between x_L and x_R. The left and right reservoirs are then defined for $x < x_L$ and $x > x_R$ respectively. We suppose that the system is driven from equilibrium by an electrostatic potential $eV(x)$, which couples to the edge charge density $\rho(x)$, and is a constant $eV_{L(R)}$ in the left (right) reservoir. The underlying physical assumption of this approach is that the edge states which emanate from a given reservoir are in equilibrium at the chemical potential of that reservoir.

Since the edge current operator is linear in the boson field, the edge current at x which flows in linear response to $V(x')$ may be computed directly. Specifically,

$$\langle I(x)\rangle = \int dx' D^R(x - x', \omega \to 0)V(x'), \tag{2.19}$$

where the retarded response function is given by

$$D^R(x - x', \omega) = -i\int_{-\infty}^{0} dt e^{-i\omega t}\frac{e^2}{(2\pi)^2\hbar} \langle[\dot{\phi}(x, 0), \partial_{x'}\phi(x', t)]\rangle. \tag{2.20}$$

This may be computed using (2.18) by analytically continuing the imaginary time response function

$$D(x - x', \omega_n) = -\frac{e^2}{h}\sum_q e^{iq(x-x')}\frac{q\omega_n}{q(\eta i\omega_n - vq)} \tag{2.21}$$

to real frequencies, $i\omega_n \to \omega + i\epsilon$. Here $\eta = \pm 1$ determines the direction of edge propagation. We find

$$D^R(x - x', \omega) = \frac{e^2}{h}\theta\left(\eta(x - x')\right)\ \frac{i\eta\omega}{v}e^{i\eta(\omega+i\epsilon)(x-x')/v}. \tag{2.22}$$

The θ function reflects the chiral nature of the edge propagation, showing that the current at x depends only on the voltages at positions x' "upstream" of x. In the limit $\omega \to 0$, the integral in (2.19) will be dominated by values of x' that are deep into the "upstream" reservoir. Thus, for $\eta = +1$, which corresponds to an edge which propagates from left to right, the current is

$$\langle I\rangle = \frac{e^2}{h}V_L. \tag{2.23}$$

The two terminal conductance of a Hall bar follows upon adding a contribution from the opposite edge which emanates from the right reservoir and contributes a current $-(e^2/h)V_R$. The net current is thus $I = G(V_L - V_R)$, with an appropriately quantized two terminal conductance: $G = e^2/h$.

It is straightforward to generalize the above approach, based on the right/left conductances, to compute the conductance measured in a four terminal geometry. One thereby reproduces the multiterminal Buttiker-Landauer[13] transport formula for non-interacting electrons.

An advantage of bosonization even for the IQHE edge is the ease in which electron interactions can be incorporated. Consider specifically short-range electron-electron interactions acting between the electrons along the edge. (In this paper, we will ignore throughout the long-ranged piece of the Coulomb interaction, assuming it to be screened by a ground plane.) The appropriate term to add to the Hamiltonian is $v_{int}\rho^2$, with ρ the electron density. This term is quartic in fermion fields, but can be simply absorbed into the velocity in the bosonized Hamiltonian (2.17). Thus electron interactions simply shift the edge velocity. They do not alter the quantized conductance, which is independent of v.

B. FQHE

In the fractional quantum Hall effect one typically has a partially filled Landau level. In the absence of electron-electron interactions there would then be an enormous ground state degeneracy, but this degeneracy is lifted by the interactions. At special filling factors, such as $\nu = 1/3$, the system is expected to condense into a correlated liquid state. This liquid state is incompressible, and has a gap for all excitations. In the presence of edges one anticipates low lying edge excitations, as in the IQHE. There are several routes to the appropriate description of the FQHE edge states. We briefly describe two. The first, heuristic in nature, involves generalizing the bosonized action of the IQHE edge so that the resulting conductance is fractional. The second, discussed in subsection 2 below, generates the equivalent edge description starting from a Landau-Ginsburg theory for the bulk Hall fluid. The latter approach can be easily generalized to hierarchical FQHE states, as described in subsection 3.

1. Heuristic motivation

In the derivation of the conductance for the IQHE edge from the bosonized action, the quantization can be traced to the prefactor of the first term in (2.18) - which is a fixed dimensionless number $(1/4\pi)$. This prefactor also determines the coefficient on the right side of the Kac-Moody commutation relation (2.14). By simply replacing the prefactor $1/4\pi$ by $1/4\pi\nu$, where ν is the filling factor, one obtains an edge action,

$$S = \frac{1}{4\pi\nu} \int dx d\tau \partial_x \phi (i\partial_\tau + v\partial_x)\phi, \tag{2.24}$$

which has the required fractional conductance:

$$G = \nu \frac{e^2}{h}. \tag{2.25}$$

This innocuous looking change has most striking consequences. The most dramatic is that the charge e excitation will generally have fractional statistics. To see this note first that the boson field now satisfies a modified commutation relation

$$[\phi(x), \phi(x')] = i\pi\nu \mathrm{sgn}(x - x'). \tag{2.26}$$

The conjugate momentum is thus no longer the edge density, but rather $\Pi = (1/\nu)\rho$ with $\rho = (1/2\pi)\partial_x\phi$ as before. The operator for a charge e edge excitation then becomes

$$\psi(x) = e^{i2\pi \int^x dx' \Pi(x')} \rightarrow e^{i\phi/\nu}. \tag{2.27}$$

Upon using the commutation relation (2.26) for the ϕ field, one deduces that the charge e operator generally has fractional statistics:

$$\psi(x)\psi(x') = e^{i\pi \mathrm{sgn}(x-x')/\nu} \psi(x')\psi(x). \tag{2.28}$$

Notice that for the special case of $1/\nu$ an odd integer, the charge e excitation is fermionic, and can then be associated with the electron.

For more general ν, the absence of a charge e fermionic edge excitation is rather worrisome. Since the bulk FQHE state is built from physical electrons, one would expect electrons to be present at the edge also. This reasoning suggests that the effective action (2.24) is only a valid description of the Hall edge, when $\nu = 1/m$ with m an odd integer[17]. This conclusion will be confirmed below, where it is shown that for Hall states with $\nu \neq 1/m$, there are multiple edge modes.

Besides the operator $\psi(x) = e^{i\phi/\nu}$ which creates a charge e fermion at the edge of the $\nu = 1/m$ fluid, one can consider other operators, such as $e^{i\phi}$. This operator creates an edge excitation with fractional charge, νe. The statistics is also fractional, with a phase factor of $exp(\pm i\nu\pi)$ under exchange. It is apparent that $e^{i\phi}$ creates a Laughlin quasiparticle at the edge. This will be confirmed below starting from a Ginsburg-Landau description of the Hall fluid.

2. From Ginzburg-Landau theory

The physical idea behind the Ginzburg-Landau theory of the FQHE[9,10] is that vortices in the electron wave function bind to the electrons. For the Laughlin sequence of states at filling $\nu = 1/m$, each electron binds with m vortices. This can be seen explicitly in Laughlin's celebrated wave function, in which m vortices sit right on top of each electron, but is expected to be a general feature of the incompressible Hall fluid. Due to the 2π phase accumulated upon encircling each vortex, the statistics of the electron/vortex composites are bosonic, since m is an odd integer. At the magic filling $\nu = 1/m$ all of the field induced vortices are accommodated by binding to electrons, and the composite bosons do not "see" any residual vortices. They can then Bose condense, forming a superfluid. This condensed fluid describes dissipationless flow and a (bulk) quantized Hall conductance, the two hallmarks of a quantum Hall fluid.

The original Ginzburg-Landau theory focussed on the bosonic wave function of the electron/vortex composite.

There is an alternate dual description, though, which consists of a bosonic field $e^{i\phi}$ which creates a vortex in the Ginzburg-Landau complex boson field[36,35]. This field is minimally coupled to a gauge field, a_μ. The gauge field is directly related to the electron three-current via

$$j_\mu = \epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda / 2\pi. \tag{2.29}$$

Thus when a vortex moves, it sees the electrons as a source of "fictitious" flux. In a quantum Hall fluid at magic filling there are no free vortices in the Ginzburg-Landau field. Vortex anti-vortex pairs can be created, but cost a finite energy and are unimportant at low temperatures. (This corresponds to a Laughlin quasiparticle and quasihole pair.) The low energy description thus reduces to that of the gauge field alone. Keeping the most important terms, the resulting Euclidean action for this gauge field is

$$S_{\rm bulk} = im\frac{1}{4\pi}\int a_\mu \epsilon_{\mu\nu\lambda}\partial_\nu a_\lambda. \tag{2.30}$$

This effective action describes the long wavelength density fluctuations of the condensed fluid. The energy gap for the bulk quasiparticle excitations is not specified. Provided the temperature is well below this gap, the effective action (2.30) provides an adequate description. However, at filling factors away from $\nu = 1/m$, there will be some residual vortices, and the electron/vortex composites can only condense if these residual vortices are pinned and localized by bulk impurities. In this case, there will be many low energy, but spatially localized, excitations involving re-arranging the positions of these vortices. The effective action (2.1) can presumably still be used to extract transport properties, though, since at low temperatures the localized vortices will not contribute significantly to the transport. (This is not the case for other physical properties such as the electronic specific heat.)

Although the vortex (quasiparticle) excitations are not important at low T in the bulk, they play a crucial role at the edge. At the edge their gap vanishes and they form the edge states. The edge excitations can be naturally expressed in terms of the phase ϕ of the vortex creation operator, $e^{i\phi}$, which is minimally coupled to the gauge field, $\partial_\mu\phi - a_\mu$. Specifically, as shown by Wen[17], the bulk degrees of freedom can be eliminated from (2.30) by integration over a_τ, which imposes an incompressibility constraint on bulk density fluctuations: $\vec{\nabla}\times\vec{a} = 0$, with the vector referring to the two spatial components. A scalar field can then be introduced to solve this constraint, $\vec{a} = \vec{\nabla}\phi$, where ϕ can be interpreted as the phase of the vortex. After an integration by parts the final Euclidean action for the edge states is given by (2.14) - the form we arrived at by heuristic argument above.

As discussed by Wen[17], the magnitude of the edge velocity v depends on the precise boundary conditions assumed for the field a_τ. Since v will depend on the edge confining potential and the edge Coulomb interaction, it cannot be determined from the bulk action (2.30). It is thus appropriate to take the velocity v as a phenomenological parameter.

It follows from (2.29) that the 1d charge density along the edge is given by $\rho = \partial_x\phi/2\pi$, precisely as in (2.12). Our identification of $e^{i\phi}$ as the creation operator for a quasiparticle (or vortex) at the edge was also appropriate. We have thereby arrived a description of the $\nu = 1/m$ edge, identical to that obtained heuristically above. While neither "derivation" is rigorous, the final effective action (2.24) is undoubtedly correct. The Ginzburg-Landau approach is advantageous, though, since it can be readily generalized to hierarchical quantum Hall fluids, as we now describe

3. Hierarchical states

Soon after Laughlin's wavefunction, Haldane and Halperin[37] suggested a way to account for Hall plateaus at other rational fillings, besides $\nu = 1/m$. Their basic idea was that the vortices introduced upon moving away from filling $\nu = 1/m$ would themselves condense into a Laughlin fluid. By successively iterating this procedure, incompressible Hall fluids at arbitrary rational ν, with odd denominator, could be generated. An alternate hierarchical construction, suggested by Jain, consists of attaching an even number, p, of flux tubes (or vortices) to each electron, and then putting the resulting fermionic electron/vortex composites into n full Landau levels. This describes states at filling $\nu = n/(np+1)$, a robust sequence of fractions. While the wavefunctions in these different constructions certainly differ in detail, it was initially unclear as to whether they describe the "same phase".

This was clarified by Wen and Zee[36] and Read[38] who emphasized a "hidden" topological order in the Hall fluids, and introduced a scheme to characterize and classify them. Specifically, at the n'th level of the Haldane/Halperin hierarchy the fluid can be characterized by a symmetric $n\times n$ matrix K. The appropriate effective action which generalizes (2.30) can be expressed in terms of n gauge fields and takes the form:

$$S_{\rm bulk} = \frac{i}{4\pi}\int a^i_\mu K_{ij}\epsilon_{\mu\nu\lambda}\partial_\nu a^j_\lambda. \tag{2.31}$$

The electron 3-current is given by

$$j_\mu = \sum_i t_i \epsilon_{\mu\nu\lambda}\partial_\nu a^i_\lambda/2\pi, \tag{2.32}$$

where t_i are basis dependent integers or "charges". This is the multi-component generalization of (2.29). The filling factor is given by

$$\nu = \sum_{ij} t_i K^{-1}_{ij} t_j. \tag{2.33}$$

The K matrix also characterizes the charge and statistics of the bulk quasiparticle excitations. Specifically, the quasiparticles are labeled by a set of integers m_j, with $j = 1, 2, .., n$, with charge (in units of the electron charge)

$$Q = \sum_{ij} m_i K_{ij}^{-1} t_j \tag{2.34}$$

and statistics angle

$$\frac{\Theta}{\pi} = \sum_{ij} m_i K_{ij}^{-1} m_j. \tag{2.35}$$

In this approach, all of the universal properties of the bulk quantum Hall state follow directly from the K matrix.

The explicit form of the K matrix for a given quantum Hall state depends on the choice of basis, as do the integers t_i[36,38]. Under a basis transformation

$$K \to K' = W^T K W, \tag{2.36}$$

$$t_i \to t_i' = W_{ij}^T t_j, \tag{2.37}$$

and

$$m_i \to m_i' = W_{ij}^T m_j, \tag{2.38}$$

where W is an n by n matrix with integer matrix elements and unit determinant. This transformation leaves the filling ν as well as the quasiparticle charge and statistics invariant. The basis in which $t_i = 1$ for all i, is called the "symmetric" basis by Wen and Zee. In the "hierarchical" basis, $t_1 = 1$ and $t_i = 0$ for $i = 2, 3, ..., n$.

The form of the K matrix obtained for a FQHE fluid at filling ν from the Haldane/Halperin hierarchy scheme is identical to that obtained via Jain's construction[40]. Thus the two states are topologically equivalent. In the "symmetric" basis, for filling $\nu = n/(np+1)$ with integer n and even integer p, the K matrix is given explicitly by

$$K_{ij} = \delta_{ij} + p. \tag{2.39}$$

At filling $\nu = 1/(p_1 - 1/p_2)$ where p_1 and p_2 are odd and even integers respectively, the K matrix in the "hierarchical" basis takes the simple form:

$$K = \begin{pmatrix} p_1 & -1 \\ -1 & p_2 \end{pmatrix}. \tag{2.40}$$

The edge excitations may be described by eliminating the bulk degrees of freedom, as described in subsection 2 above. Upon integration over a_τ^i, a constraint on the density fluctuations in the bulk is imposed: $\vec{\nabla} \times \vec{a}^i = 0$, for all $i = 1, 2, ..., n$. Scalar fields can then be introduced to solve these constraints, $\vec{a}^i = \vec{\nabla}\phi_i$, one for each gauge field. As above, the edge excitations are described in terms of these scalar fields. The appropriate effective action at the edge can then be written as $S = S_0 + S_1$ with

$$S_0 = \int dx d\tau \frac{1}{4\pi} \sum_{ij} (\partial_x \phi_i) K_{ij} (i\partial_\tau \phi_j) \tag{2.41}$$

and

$$S_1 = \int dx d\tau \frac{1}{4\pi} \sum_{ij} V_{ij} \partial_x \phi_i \partial_x \phi_j. \tag{2.42}$$

The first term is solely determined from bulk physics of the Hall fluid. This term determines the commutation relations for the bose fields:

$$[\phi_i(x), \phi_j(x')] = i\pi K_{ij}^{-1} \mathrm{sgn}(x - x'), \tag{2.43}$$

the generalization of (2.26). In addition we also have interaction terms of the form $\partial_x \phi_i \partial_x \phi_j$. These interaction strengths are non-universal, depending on the edge confining potential and edge electron interactions.

It follows from (2.32) that the edge charge density is given by

$$\rho(x) = \frac{1}{2\pi} \sum_{i=1}^{n} t_i \partial_x \phi_i. \tag{2.44}$$

Operators which create charge at the edge can be deduced from the conjugate momentum: $\Pi_i = (1/2\pi) K_{ij} \partial_x \phi_j$. An operator of the form $\exp i\phi_i(x)$ creates "instantons" in the boson fields ϕ_j at position x. These instantons carry charge, as seen from (2.44). The general operator

$$\hat{T}(x) = e^{i \sum_{j=1}^n m_j \phi_j(x)} \tag{2.45}$$

for integer m_j, creates an edge excitation at x with charge Q given by (2.34).

Since the effective action (2.41)-(2.42) is quadratic in the boson fields, all physical quantities can be readily computed. Unfortunately, when the eigenvalues of K are not all of the same sign, the computed conductance is not given[23,25] by the quantized value $|\nu| e^2/h$. To see why consider for simplicity a two channel case such as $\nu = 2/3$. Since the sign of the eigenvalues of the K matrix determine the direction of propagation, the two modes will be moving in opposite directions, as sketched in Fig. 5. Consider a two-terminal transport situation in which the source and drain are held at different chemical potentials. Since an edge mode is, by assumption, in equilibrium with the reservoir from which it emanates, the current on the top edge will depend on the voltages in both reservoirs. Specifically, the current at the top edge can be written

$$I_{top} = \frac{e^2}{h} \left(g_+ V_L - g_- V_R \right), \tag{2.46}$$

where g_+ (g_-) is the dimensionless conductance for the right (left) moving channel. This form can be derived explicitly by generalizing (2.19) to include multiple edge modes.

The two terminal conductance follows by considering the other edge, which carries a current $g_- V_L - g_+ V_R$, giving

$$G = \frac{e^2}{h}(g_+ + g_-). \tag{2.47}$$

Notice that the conductances add, even though the channels are moving in opposite directions. Using (2.19) and the action (2.41)-(2.42) one can explicitly compute g_+ and g_-. Their difference is found to be universal and quantized, $g_+ - g_- = \nu$. However, their sum, which enters in the conductance, is non universal, depending on the interaction strengths V_{ij} in (2.42).

It is straightforward to generalize this approach to compute the conductance measured in a four terminal geometry. In particular, we find that the four terminal Hall conductance is given by

$$G_H = \frac{e^2}{h}\frac{g_+^2 + g_-^2}{g_+ - g_-}. \tag{2.48}$$

It is only when all channels propagate in the same direction that G_H is universal and equal to $\nu e^2/h$.

The non-quantized conductance for hierarchical states with multiple modes which move in both directions, is in glaring contradiction with experiment. Clearly some important physics must be absent from the simple effective action (2.41)-(2.42). A clue can be seen from Fig. 5, where it is clear that in a transport situation, right moving edge modes are in equilibrium with the left reservoir, and left movers in equilibrium with the right reservoir. Thus in the presence of a non-zero source-to-drain voltage, opposite moving edge modes on a given edge will be out of equilibrium with one another. But since these modes are in close proximity, what stops them from equilibrating?

In the effective action (2.41)-(2.42) there are no terms which transfer charge between the different edge modes, to allow for possible equilibration. But surely in real experimental systems there will be equilibration processes present. A stringent constraint is that charge transfer must conserve energy and momentum. Generally, different edge modes have different momenta - the gauge invariant momentum difference between two modes being proportional to the magnetic flux threading the space between them. Since the edge modes are all at the same energy (in equilibrium), charge transfer processes with the emission of phonons or photons to take up the momentum, will not conserve overall energy. These processes are thus forbidden.

However, if there are impurities near the edge, momentum of the edge modes need not be conserved. Momentum can be transferred to the center of mass of the sample, through the impurities. Thus impurity scattering at the edge will mix the different modes and tend to equilibrate them. In Section 4 we study this effect in detail.

III. RANDOMNESS AND HIERARCHICAL EDGE STATES

A. Introduction

In section II we argued that for hierarchical states with multiple edge branches moving in both directions, the conductance is predicted to be non-universal due to an absence of inter-mode equilibration. Edge impurity scattering is then critical, to allow for equilibration, and must be incorporated into a realistic edge state theory. We now consider the general problem of random impurity scattering at the edge of a quantum Hall state. For a broad class of hierarchical states, we will show that the low temperature physics is described by a new random fixed point.

In contrast to the hierarchical states, for Laughlin states which have a single edge channel disorder is unimportant. To see this consider first a $\nu = 1$ edge with a spatially dependent random edge potential $\mu(x)$. Treating the electrons as non-interacting, we may add this random potential to the fermion action (2.10):

$$S = \int dx d\tau \psi^*(\partial_\tau + iv\partial_x)\psi + \mu(x)\psi^*\psi. \tag{3.1}$$

The random term can be eliminated by performing a spatially dependent $U(1)$ gauge transformation, $\tilde{\psi} = \psi \exp i\delta(x)$, with

$$\delta(x) = \frac{1}{v}\int_{-\infty}^{x} dx' \mu(x'). \tag{3.2}$$

Thus, the only effect of the random potential is to introduce an unimportant forward scattering phase shift, $\delta(x)$.

This result may be extended to interacting systems and to Laughlin states by using the chiral boson representation (2.24). In this representation, the random potential $\mu(x)$ couples to $\rho(x) = \partial_x\phi/2\pi$, and may be eliminated via the transformation $\tilde{\phi}(x) = \phi(x) + \nu\delta(x)$.

Clearly, any non trivial effects of edge disorder must arise from tunneling between different channels. To study such effects, we consider first the IQHE with $\nu = 2$ in detail. In doing so we will develop the necessary machinery to describe hierarchical FQHE states. We will then focus on three specific cases, $\nu = 2/3$, $\nu = 2/5$ and $\nu = 4/5$. As emphasized in section II the case $\nu = 2/3$ is particularly important, since the model without disorder predicts a non quantized conductance.

B. $\nu = 2$ Random Edge

1. Fermion Representation

Consider first the simplest possible two channel model: non interacting electrons with $\nu = 2$, where the two edge modes are identical. In terms of a two component chiral fermion field, $\Psi = (\psi_1, \psi_2)$, the appropriate action generalizing (2.10) is,

$$S_0 = \int dx d\tau \Psi^\dagger (\partial_\tau + iv\partial_x)\Psi. \qquad (3.3)$$

Note that S_0 is invariant under a global $U(2)$ transformation. This symmetry is a product of a $U(1)$ symmetry, arising from conservation of charge, and an $SU(2)$ symmetry arising from the conservation of "spin". As we shall see below, this high symmetry, which appears rather artificial since it requires the channels to be identical, is actually a generic property of the groundstate when impurity scattering is present.

Consider now the effect of randomness. In addition to random potentials coupling to the charge densities in each channel, randomness will also introduce tunneling between the two channels. In general we may write,

$$S_{\rm random} = \int dx d\tau \Psi^\dagger \left[\mu(x) + \vec{\xi}(x)\cdot\vec{\sigma}\right]\Psi, \qquad (3.4)$$

where $\vec{\sigma}$ are Pauli matrices. Here $\mu(x) \pm \xi_z(x)$ are random potentials coupling separately to the two channels. The spatially random coefficients, $\xi_\pm = \xi_x \pm i\xi_y$, multiply $\psi_1^\dagger\psi_2$ and $\psi_2^\dagger\psi_1$, and correspond to inter-channel tunneling.

As in the single channel case, $\mu(x)$ may be eliminated by the $U(1)$ gauge transformation (3.2). Similarly, $\vec{\xi}(x)$ may be eliminated by an $SU(2)$ rotation. We thus write

$$\tilde{\Psi}(x) = e^{i\delta(x)}U(x)\Psi(x), \qquad (3.5)$$

where δ is given by (3.2) and and $U(x)$ is an $SU(2)$ rotation, given by

$$U(x) = T_x \exp\left[\frac{i}{v}\int_{-\infty}^{x} dx' \vec{\xi}(x')\cdot\vec{\sigma}\right], \qquad (3.6)$$

where T_x is an x-ordering operator. The resulting random problem is then given by

$$S_0 + S_{\rm random} = \int dx d\tau \tilde{\Psi}^\dagger (\partial_\tau + iv\partial_x)\tilde{\Psi}. \qquad (3.7)$$

In terms of the new field $\tilde{\Psi}$, the random problem still has an exact global U(2) symmetry.

The exact solution (3.7) gives a complete description of the non-interacting problem. The eigenstates are simply plane waves in which the channel index (or "spin") is rotated by $U(x)$. The strength of the random interchannel tunneling introduces a mean free path, ℓ, which is the length scale over which $U(x)$ varies. On length scales longer than ℓ, $e^{i\delta}U(x)$ will be an uncorrelated random $U(2)$ matrix.

Wen has generalized the above solution to allow for different velocities of the two channels[39]. In this case, the $U(2)$ symmetry of the clean edge is broken. Nevertheless, on length scales longer than the mean free path ℓ, all excitations move at a single velocity: $\bar{v} = 2v_1v_2/(v_1+v_2)$. This is physically reasonable, since upon averaging over long times an electron spends equal time in each channel, regardless of it's initial channel. The long distance behavior is thus argued by Wen to be the same as that of the $U(2)$ symmetric model (3.3) in which the velocities are equal.

2. Boson Representation

The above non-interacting electron theory cannot be easily generalized to include the effects of interactions, and moreover is not suited to the FQHE. It is therefore desirable to reformulate the random edge in a chiral boson representation. In doing so, we shall establish that the low energy physics of the random $\nu = 2$ edge is described by a stable random fixed point. However, in contrast to the non interacting problem (3.7), in which both modes move at the same velocity, the interactions play an important role by giving rise to "spin-charge" separation, in which the charged and neutral modes propagate at different velocities. The final fixed point has a U(1)xSU(2) symmetry, rather than the full U(2) symmetry present in the non-interacting case (3.7).

Using (2.18) the bosonized action for a $\nu = 2$ edge can be written as,

$$S = S_0 + S_{\rm random} \qquad (3.8)$$

with,

$$S_0 = \int dx d\tau \frac{1}{4\pi}[\partial_x\phi_1(i\partial_\tau + v_1\partial_x)\phi_1 + \partial_x\phi_2(i\partial_\tau + v_2\partial_x)\phi_2 + 2v_{12}\partial_x\phi_1\partial_x\phi_2] \qquad (3.9)$$

Electron interactions between the two channels are accounted for by v_{12}. By using (2.16), which expresses the electron operator as $e^{i\phi}$, the interchannel tunneling may be written

$$S_{\rm random} = \int dx d\tau \left[\xi_+(x)e^{i(\phi_1-\phi_2)} + \xi_-(x)e^{-i(\phi_1-\phi_2)}\right]. \qquad (3.10)$$

As in (3.4) $\xi_\pm(x)$ are spatially random complex tunneling amplitudes. We omit the random potentials $\mu(x)$ and $\xi_z(x)$, which couple to $\partial_x(\phi_1 \pm \phi_2)$, since they may be eliminated via a suitable redefinition of $\phi_{1,2}$.

Since the operators in (3.10) are non-linear in the boson fields, the random model appears rather intractable. One approach is to expand for small $\xi_\pm(x)$ about the free theory, S_0. This is problematic, however, because the perturbation theory is divergent at low energies. One can nevertheless define a perturbative renormalization group (RG) transformation[23,25] in powers of the variance, W, defined via $[\xi_+(x)\xi_-(0)]_{ens} = W\delta(x)$. Here the square brackets denote an ensemble average over realizations of the disorder. As outlined in appendix A, the leading order RG flow equations take the form

$$\frac{\partial W}{\partial \ell} = (3 - 2\Delta)W \tag{3.11}$$

where Δ is the scaling dimension of the operator $\hat{O} = \exp(i(\phi_1 - \phi_2))$ evaluated in the free theory, S_0. One finds $\Delta = 1$, which also follows from the fermion representation (3.3), since $\hat{O}$ is the product of two fermion operators. From (3.11) it is clear that the random potential is relevant, so that non perturbative methods are necessary. Fortunately, the low energy physics is controlled by a stable fixed point which is accessible non-perturbatively, as we now show.

To proceed it is convenient to introduce charge and "spin" variables[23],

$$\phi_\rho = \phi_1 + \phi_2 \tag{3.12}$$

$$\phi_\sigma = \phi_1 - \phi_2. \tag{3.13}$$

The net charge density on the edge is $\partial_x \phi_\rho / 2\pi$. The field ϕ_σ, which commutes with ϕ_ρ, represents the neutral degree of freedom associated with tunneling between the channels. In terms of these new variables we may rewrite (3.8) as

$$S = S_\rho + S_\sigma + S_{\rm int} \tag{3.14}$$

with charge and neutral pieces

$$S_\rho = \frac{1}{8\pi} \int dx d\tau \partial_x \phi_\rho (i\partial_\tau + v_\rho \partial_x)\phi_\rho, \tag{3.15}$$

$$S_\sigma = \int dx d\tau [\frac{1}{8\pi} \partial_x \phi_\sigma (i\partial_\tau + v_\sigma \partial_x)\phi_\sigma + \xi_+(x) e^{i\phi_\sigma} + c.c.], \tag{3.16}$$

coupled together via

$$S_{\rm int} = \frac{v_{\rm int}}{4\pi} \int dx d\tau \partial_x \phi_\rho \partial_x \phi_\sigma. \tag{3.17}$$

The velocities v_ρ, v_σ and v_{int} depend on the original velocities in (3.9). Note that v_ρ is not necessarily equal to v_σ. Indeed, a short range interaction which couples to the total charge affects only v_ρ.

Consider first the $U(2)$ symmetric model, as considered in (3.3), with $v_1 = v_2 = v$, $v_{12} = 0$. Upon transforming to the charge-spin variables, we then find $v_\rho = v_\sigma = v$ and $v_{\rm int} = 0$. The system thus decouples into a "charge" sector described by S_ρ and a "spin" sector S_σ. Note that the interchannel tunneling terms only affect the spin sector. Despite the presence of complicated nonlinear random terms in S_σ, we know that this model is equivalent to the exactly soluble fermion model (3.7). The hidden simplicity in S_σ is a consequence of the fact that the operators $\cos\phi_\sigma$, $\sin\phi_\sigma$ and $\partial_x\phi_\sigma$ transform as S_x, S_y and S_z in an $SU(2)$ algebra, known as a level-one $SU(2)$ current algebra. It is this connection that will allow us to formulate a general solution of the interacting problem.

Consider now the case in which $v_{\rm int} = 0$, but $v_\rho \neq v_\sigma$. This no longer corresponds to noninteracting electrons. The charge and neutral sectors are still decoupled, but now the charge and neutral modes propagate at different velocities. Again, the only non trivial part of the problem is the neutral sector, which involves nonlinear random terms. But the neutral sector is *identical* to the neutral sector of the exactly soluble problem (3.3) provided we identify $v = v_\sigma$. Since the soluble fermion problem includes a charge sector, in order to take advantage of this connection we must introduce an auxiliary field $\tilde{\phi}_\rho$ with an action $\tilde{S}_\rho$ which is the same as S_ρ, except v_ρ is replaced by v_σ. The field $\tilde{\phi}_\rho$ does not enter physical observables, but allows a convenient representation of the $SU(2)$ symmetry in the neutral sector. The combined action $\tilde{S}_\rho + S_\sigma$ may be "fermionized", by letting $\psi_1 = \exp[i(\tilde{\phi}_\rho + \phi_\sigma)/2]$ and $\psi_2 = \exp[i(\tilde{\phi}_\rho - \phi_\sigma)/2]$. The operator corresponding to $\partial_x\phi_\sigma$ is $\Psi^\dagger\sigma_z\Psi$. The resulting fermion problem is identical to (3.4), and may be solved by the space dependent transformation $\tilde{\Psi} = U(x)\Psi$ with $U(x)$ given in (3.6). The action is then simply by (3.7) with $v = v_\sigma$. This form exposes the hidden but exact global SU(2) symmetry of the neutral sector (3.16).

Thus, we see that when $v_{\rm int} = 0$, the problem is exactly soluble even in the presence of interactions, and the ground state has an exact $U(1) \times SU(2)$ symmetry described by (3.15) and (3.7). However, in contrast to the non interacting case the charge and neutral modes need not move at the same velocity: there is "charge-spin separation". We now must study the effects of $v_{\rm int}$, which couples the charge and neutral sectors, and hence breaks the $SU(2)$ symmetry. Using the above exact solution we will show that in the presence of a random potential, $v_{\rm int}$ is *irrelevant*. Under an RG transformation, the system scales to the charge-neutral decoupled fixed point.

It is convenient to re-express $S_{\rm int}$ in terms of the fermion field $\tilde{\Psi}$. We find

$$S_{\rm int} = \frac{v_{\rm int}}{4\pi} \int dx d\tau \partial_x \phi_\rho \tilde{\Psi}^\dagger U(x) \sigma_z U^\dagger(x) \tilde{\Psi}. \tag{3.18}$$

The relevancy of this term may be deduced from the scaling dimension of the operator $\hat{O}_{ij} = \partial_x \phi_\rho \tilde{\Psi}_i^\dagger \tilde{\Psi}_j$, which we denote δ_v. Using (3.17) and (3.15) one readily obtains $\delta_v = 2$. Since this operator has a random x dependent coefficient $v(x) \approx v_{int} U \sigma_z U^\dagger$, which is uncorrelated on spatial scales large compared to the mean free path ℓ, it is most useful to consider the scaling behavior of the mean square average, $W_v = [v(x)^2]_{ens}$, where the square brackets denote an ensemble average. As outlined in appendix A, to leading order in W_v the RG flow equation is

$$\frac{dW_v}{d\ell} = (3 - 2\delta_v) W_v. \tag{3.19}$$

Since $\delta_v = 2$ the perturbation is irrelevant. It should be emphasized that in the absence of randomness, the dimension 2 operators in $S_{\rm int}$ are marginal and do not renormalize to zero! Thus disorder is seen to be absolutely critical in the stability of the decoupled U(1)xSU(2) fixed point.

In summary, impurity scattering has played a crucial role in driving the charge/neutral coupling to zero. The final fixed point theory has a global U(1)xSU(2) symmetry, a much higher symmetry than the underlying random Hamiltonian. While it is difficult to compute the fully renormalized velocities v_ρ and v_σ, it is clear that they will be equal only in the absence of interactions. Thus, generically, we expect "charge-spin" separation on the $\nu = 2$ edge.

C. Fractional Quantum Hall Random Edge

Using the results of the previous section for $\nu = 2$, we are now in a position to analyze the effects of impurity scattering on hierarchical FQHE edges. We will initially focus on Hall states at the second level of the Haldane-Halperin hierarchy, with $\nu = 1/(p_1 - 1/p_2)$, where p_1 and p_2 are odd and even integers respectively. We will consider three specific examples which display different generic behavior : $\nu = 2/3$, $2/5$ and $4/5$. We will then discuss generalizations to other fractions, including those at higher levels in the hierarchy.

At filling $\nu^{-1} = p_1 - 1/p_2$ the edge consists of two modes, as described in section II. Without impurity scattering the edge can be described by the action (2.41)-(2.42) with K an appropriate 2x2 matrix. In the hierarchy basis the explicit form for K is given in (2.40). As in the preceding section, it will be useful to represent the problem in terms of charge and neutral (or "spin") variables. These variables are related to the "hierarchy basis" variables by

$$\phi_\rho = \phi_1 \qquad (3.20)$$

$$\phi_\sigma = \phi_1 - p_2\phi_2. \qquad (3.21)$$

The charge and neutral fields commute with one another, as can be seen using (2.43). The action (2.41)-(2.42) may then be expressed in the form

$$S = S_\rho + S_\sigma + S_{\rm int} \qquad (3.22)$$

with charge and neutral pieces

$$S_\rho = \frac{1}{4\pi\nu}\int dx d\tau \partial_x\phi_\rho(i\partial_\tau + v_\rho\partial_x)\phi_\rho \qquad (3.23)$$

$$S_\sigma = \frac{1}{4\pi p_2}\int dx d\tau \partial_x\phi_\sigma(i\partial_\tau + v_\sigma {\rm sgn}(p_2)\partial_x)\phi_\sigma \qquad (3.24)$$

coupled together via

$$S_{\rm int} = \frac{v_{\rm int}}{4\pi}\int dx d\tau \partial_x\phi_\rho\partial_x\phi_\sigma. \qquad (3.25)$$

Again, the velocities v_ρ, v_σ and $v_{\rm int}$ depend on the original velocities, V_{ij} in (2.42).

The most important terms generated by a random edge potential will be tunneling terms, as in (3.10), which transfer charge between the two channels. Charge conservation dictates that such processes do *not* create a net charge, so $Q = 0$ in (2.34). The most relevant such term is given by $\exp \pm i(\phi_1 - p_2\phi_2) = \exp i\phi_\sigma$. We are thus lead to consider,

$$S_{\rm random} = \int dx d\tau \left[\xi_+(x)e^{i\phi_\sigma} + c.c.\right]. \qquad (3.26)$$

Note that the random tunneling operators only involve the neutral field, ϕ_σ.

We now consider three specific examples which display different types of generic behavior.

1. $\nu = 2/3$

The $\nu = 2/3$ edge is described by $p_1 = 1$ and $p_2 = -2$. Consider first an impurity free edge with $S_{\rm random} = 0$. In this case, when $v_{\rm int} = 0$ the charge and neutral modes decouple, and the neutral mode propagates in the direction opposite to the charge mode. Since the conductance depends only on the charge mode, it may be seen using (2.25) that it is appropriately quantized, $G = (2/3)e^2/h$. However, when $v_{\rm int} \neq 0$, the conductance will be non-quantized and non-universal, depending on $v_{\rm int}$. This can be seen explicitly using (2.19). The two modes continue to propagate in different directions, but both contribute to the conductance. In general we may write $G = (2/3)\Delta e^2/h$, where $\Delta > 1$. Then $\Delta = 1$ only when $v_{\rm int} = 0$.

We shall now argue, that as in the $\nu = 2$ case, the presence of a random edge potential can drive the system to the decoupled fixed point with $v_{\rm int} = 0$. Thus the conductance is quantized even when channels move in opposite directions.

Consider again the decoupled point, $v_{\rm int} = 0$. Since $p_2 = -2$, the neutral sector (3.24) and (3.26) is identical to the neutral sector (3.16) for $\nu = 2$ studied in the previous section (up to a sign which determines the direction of propagation). Thus, the exact solution obtained there by re-fermionizing the problem may be applied. Moreover, we may use exactly the same arguments to show that $v_{\rm int}$ is irrelevant. We thus establish that the decoupled fixed line is stable. The fixed point is characterized by an exact $SU(2) \times U(1)$ symmetry.

Having established the stability of the decoupled fixed line, we must also consider the possibility of other stable fixed points which could describe different phases. Consider treating the randomness (3.26) perturbatively. As in (3.11), we may analyze the relevancy of weak disorder by considering the leading order RG for the variance, W, of $\xi_\pm$:

$$\frac{dW}{d\ell} = (3 - 2\Delta)W. \tag{3.27}$$

Here Δ is the scaling dimension of the operator $\exp i\phi_\sigma$, and may be computed explicitly using the action (3.23)-(3.25). When $v_{\rm int} = 0$, then $\Delta = 1$, but $\Delta > 1$ is nonuniversal when $v_{int} \neq 0$. Indeed, Δ is the same quantity which entered into the conductance above. As seen from (3.27) when $v_{\rm int}$ is tuned so that Δ exceeds $3/2$, there will be an edge phase transition to a phase in which (weak) disorder is irrelevant. For filling $\nu = 2/3$ this transition was analyzed in Ref. 23. There it was shown that the transition is of Kosterlitz-Thouless type, with the RG flows shown in Fig. 6.

We thus conclude that there are two possible phases for the random $\nu = 2/3$ edge. For sufficiently large $v_{\rm int}$ there is a phase in which disorder is irrelevant. At zero temperature, this phase is characterized by a nonuniversal conductance, and nonuniversal tunneling exponents as shown in Section IV. The more generic phase, favored by electron interactions is characterized by charge spin separation, an exact $U(1)\times SU(2)$ symmetry, and a quantized conductance. Unlike the case $\nu = 2$, however, the charge and neutral (or "spin") modes propagate in opposite directions.

2. $\nu = 2/5$

The $\nu = 2/5$ edge is described by $p_1 = 3$ and $p_2 = 2$. In this case, since $p_2 > 0$, both modes propagate in the same direction, as seen from (3.24). Thus, as shown in Section II, even in the absence of edge impurity scattering the conductance is quantized: $G = (2/5)e^2/h$. Nonetheless, when impurity scattering is present the edge re-structures and exhibits spin-charge separation. To see this simply repeat the argument for $\nu = 2/3$, which shows that the charge and neutral sectors decouple at low energies, $v_{\rm int} \to 0$. The fixed point with U(1)xSU(2) symmetry is stable. Both the charge and neutral modes move in the same direction, generally with different velocities. Since the scaling dimension of the tunneling operator in (3.26) is $\Delta = 1$ for any interaction strength, the disorder free edge is always perturbatively unstable to impurity scattering, in contrast to the $\nu = 2/3$ case.

3. $\nu = 4/5$

The Hall fluid with $\nu = 4/5$ is described by $p_1 = 1$ and $p_2 = -4$, and the two edge modes move in opposite directions. In this case, since $|p_2| \neq 2$, the neutral sector is no longer identical to the neutral sector of $\nu = 2$, so the exact solution employed there can no longer be used. Weak impurity scattering can be analyzed perturbatively, however, by computing the scaling dimension of the tunneling operators (3.26) in the clean theory. When $v_{\rm int} = 0$ it can be shown that $\Delta = 2$. Including $v_{\rm int}$ we find $\Delta > 2$. Thus $\Delta > 3/2$, and from (3.27) weak impurity scattering is always irrelevant. Thus the only low energy fixed point is the clean edge, (3.23)-(3.25) Since the channels move in opposite directions, the zero temperature conductance is thus predicted to be non universal. At finite temperatures, however, as we see in subsection D, quantization is restored.

4. SU(n) Generalizations

We have shown that the edges of disordered $\nu = 2/3$ and $2/5$ fluids are described by a stable $T = 0$ fixed point, with an exact $U(1) \times SU(2)$ symmetry. This will also be the case for any state with $|p_2| = 2$. These filling factors can be written as $\nu = 2/(2p \pm 1)$, with p an even integer. Within Jain's construction[40], these are precisely the states which can be obtained by attaching p flux quanta to each electron and filling two Landau levels. This suggests that the above results can be generalized to the quantum Hall states derivable from n full Landau levels, which occur at filling factors $\nu = n/(np + 1)$.

This case was studied in detail in reference 25, where it was shown that a random potential drives the edge to a stable fixed point characterized by an exact $U(1)\times SU(n)$ symmetry. Again, the stable fixed point is characterized by "spin - charge" separation, but now with a more general $SU(n)$ spin. In this case, there is a single charged mode, and $n - 1$ neutral modes. The neutral modes are related by the exact $SU(n)$ symmetry, and it follows that they all move at the same velocity. In general, the charge mode moves at a different velocity, and when $p < 0$ it moves in the opposite direction of the $n - 1$ neutral modes. Only the charge mode contributes to the conductance, which is appropriately quantized, $G = \nu e^2/h$.

D. Finite Temperature Effects

The exact solution of the random edge which describes a stable zero temperature fixed point can also be used to extract physical properties of the edge at low but non-zero temperatures. These properties will be determined by the structure of the fixed point itself, and the leading irrelevant operators, such as v_{int} in (3.18). At low but non-zero temperatures these operators have not had "time" to fully renormalize to zero, and can then have an important effect on physical observables. Although one can show that the irrelevant operators do not modify the quantized Hall conductance itself, they do dramatically effect the propagation of the neutral modes at finite temperature.

To see why, we first note that the existence of the propagating neutral modes is tied intimately to the exact SU(n) symmetry in the neutral sector at the fixed point. But at finite temperatures, this symmetry is no longer exact, due to the presence of irrelevant operators, so that the neutral modes should no longer be strictly conserved. Thus, one expects that at finite temperatures the neutral modes should decay away at a non-vanishing rate, $1/\tau_\sigma$. Equivalently, one expects a finite decay length, or "inelastic scattering length", $\ell_\sigma = v_\sigma \tau_\sigma$. On scales L much larger than ℓ_σ, the neutral modes should not propagate. Since the fixed point is approached as $T \to 0$, however, the decay length should diverge in this limit.

By analyzing the leading irrelevant operators (such as (3.25)) which control the flows into the zero temperature fixed point, it was shown in Ref. 25 that the decay rate vanishes algebraically with temperature:

$$\frac{1}{\tau_\sigma} \propto T^2. \tag{3.28}$$

In contrast, the charge mode cannot decay, even at finite temperature, since electric charge is always conserved. However, due to irrelevant operators, such as (3.25), which couple the charge and neutral sectors, the charge mode can scatter off the neutral modes. This leads to a charge mode which propagates with a dispersion $\omega = v_\rho q + iDq^2$, with a "diffusion" constant D which is temperature independent at low temperatures. This implies a diffusive spreading of a charge pulse as it propagates along an edge.

For $\nu = 4/5$ we arrived at the striking conclusion that impurity scattering at the edge was ineffective at equilibrating the two edge modes, leading to a non-quantized conductance at $T = 0$. But a quantized plateau is seen at $\nu = 4/5$, albeit with less vigor than might have been expected given estimates for the bulk energy gap. This apparent conflict is resolved when one considers finite temperature effects at the $\nu = 4/5$ edge. Although disorder, W, is formally irrelevant, since $\Delta > 2$ in (3.27), at finite temperatures W has not had "time" to scale all the way to zero. In fact, by cutting off the RG flows with temperature, it was shown in Ref. 25 that there is a characteristic inelastic scattering length, which diverges at low temperatures as

$$\ell^{-1} \propto W T^{2\Delta - 2}. \tag{3.29}$$

On scales longer than this length, equilibration takes place and charge does *not* propagate upstream. Provided this length is shorter than the distance between sample probes, a quantized conductance is recovered, as discussed in Ref. 25. However, this does raise the interesting possibility of observing deviations from quantization in short Hall bars at $\nu = 4/5$ and low temperatures. A very clean sample would be favorable for observing such deviations.

IV. TUNNELING AS A PROBE OF EDGE STATE STRUCTURE

The rich physics "hidden" at the edge of FQHE fluids is not easily revealed via bulk transport measurements. An ideal way to access this edge physics is via laterally confined samples in which two edges of a given sample are brought into close proximity, allowing for inter-edge tunneling. The simplest situation is a point contact in an otherwise bulk quantum Hall fluid, as depicted in the Fig. 2. The point contact can be controlled electrostatically by a gate voltage. When the constriction is open, the two terminal conductance is given by its quantized value. However, as the channel is pinched off, and the top and bottom edges are brought into close proximity, charge will begin to backscatter between the right and left moving edge channels. Such backscattering reduces the two terminal conductance. Ultimately, as the gate voltage is increased, the Hall bar will be completely pinched off, and the two terminal conductance will be zero.

In Fig. 7, the two terminal conductance as a function of gate voltage is shown for a GaAs quantum Hall point contact[29] taken at 42mK. The two curves are taken at magnetic fields which correspond to $\nu = 1$ and $\nu = 1/3$ plateaus. To the right the point contact is open, and the conductance is quantized, whereas to the left the point contact is pinched off. Between there are many resonant structures resulting from random impurities in the vicinity of the point contact. Note the qualitative difference between the behavior for $\nu = 1$ and $\nu = 1/3$. For $\nu = 1/3$, the valleys between the resonances are deeper and the resonances are sharper.

How are we to understand this qualitative difference? In this section we present a theory of tunneling and resonant tunneling at a point contact which answers this question. We begin in IV.A with a discussion of tunneling at a point contact. We show that, in contrast to the IQHE, the conductance of a FQHE point contact *vanishes* in the limit of zero temperature. We study resonant tunneling in section IV.B, where we show that in the fractional quantum Hall effect resonances have a temperature dependent width and a universal, non Lorentzian lineshape at low temperatures. Finally, in section IV.C, we discuss low frequency shot noise at a quantum Hall point contact and suggest a method for the direct observation of the fractional charge of the Laughlin quasiparticle. We will confine our attention initially to the Laughlin states, $\nu = 1/m$, for which the edge states have a single branch. A discussion of tunneling in hierarchical quantum Hall states will be deferred to subsection C.

A. Tunneling at a Point Contact

A point contact in an IQHE fluid at $\nu = 1$, is isomorphic to a barrier in a 1d non-interacting electron gas. As discussed in Section II, the right and left moving edge states, which feed the point contact, are equivalent to the right and left Fermi points of a 1d non-interacting electron gas. According to Landauer-Buttiker transport theory[11], the two-terminal conductance through the point contact is proportional to the transmission probability, $|t|^2$, for an incident wave - the edge state - to propagate through the point contact:

$$G = \frac{e^2}{h}|t|^2. \qquad (4.1)$$

For an IQHE state at $\nu = n$, the transmission will involve the transmission probabilities of all n of the edge channels.

For a FQHE fluid at $\nu = 1/m$ a point contact is isomorphic to a barrier in a 1d interacting electron gas - a Luttinger liquid. How is the relation (4.1) for the point contact conductance modified in this case? To answer this question we use the chiral boson description of FQHE edges described in Section II. The effects of the point contact can be analyzed perturbatively in the two limits depicted schematically in Fig. 8a,b: (1) a pinched off channel with weak tunneling, and (2) an open channel with weak backscattering. These limits will be discussed in subsections 1 and 2, respectively. In subsection 3 we shall piece these two limits together into a unified description.

1. *Weak Tunneling Limit*

Consider a point contact which is almost completely pinched off. As shown in Fig. 8a, this may be described by quantum Hall fluids on the left and right hand sides, which are coupled by a weak perturbation which tunnels electrons between them. For $\nu = 1/m$ the low energy physics will be described by an edge state model of the form,

$$S = S_L^0 + S_R^0 + S_{\text{tun.}}, \qquad (4.2)$$

where the left and right edges are described by

$$S_a^0 = \frac{m}{4\pi}\int dx_a d\tau \partial_x\phi_a(i\partial_\tau + \partial_x)\phi_a, \qquad (4.3)$$

with $a = L, R$. The tunneling between these two edge states at the point contact can be expressed in terms of the edge creation and annihilation operators, and has the form

$$S_{\text{tun.}} = \int d\tau t e^{im(\phi_L - \phi_R)} + c.c. \qquad (4.4)$$

where ϕ_a is evaluated at the point contact, $x_a = 0$. Here t is the amplitude for the tunneling process (not to be confused with real time, which appears in (4.8) below).

The two-terminal conductance through the point contact can now be computed perturbatively for small tunneling amplitude t. In the presence of a voltage V across the junction, the tunneling rate to leading order can be obtained from Fermi's Golden rule:

$$I = \frac{2\pi e}{\hbar}\sum_n s_n |\langle n|H_{\text{tun.}}|0\rangle|^2 \delta(E_n - E_0 - s_n eV). \qquad (4.5)$$

Here $H_{\text{tun.}}$ is the tunneling Hamiltonian corresponding to (4.4). The sum on n is over many body states in which an electron has been transferred across the junction in the $s_n = \pm 1$ direction. It is straightforward to re-express this as

$$I = \frac{et^2}{2\pi\hbar}\int dE \left[G_L^>(E)G_R^<(E - eV) - G_L^<(E - eV)G_R^>(E)\right] \qquad (4.6)$$

where $G_a^>$ and $G_a^<$ are (local) tunneling in and tunneling out densities of states for the edge modes, related by $G^<(E) = G^>(-E)$. These can be expressed as,

$$G_a^>(E) = 2\pi\sum_n |\langle n|e^{im\phi_a}|0\rangle|^2\delta(E_n - E_0 - E) \qquad (4.7)$$

$$= \int dt e^{iEt}\langle e^{im\phi_a(t)}e^{-im\phi_a(0)}\rangle, \qquad (4.8)$$

where ϕ_a is evaluated at $x = 0$. The tunneling density of states is related to the imaginary time Green's function,

$$G(\tau) = \langle T_\tau[e^{im(\phi(\tau)-\phi(0))}]\rangle \qquad (4.9)$$

via analytic continuation, $G^>(t) = G(\tau \to it)$. Since the Euclidean action (4.3) is quadratic $G(\tau)$ may be readily computed, giving

$$G(\tau) = \left(\frac{\tau_c}{|\tau| + \tau_c}\right)^m, \qquad (4.10)$$

where τ_c is a short time cutoff. Upon analytic continuation and Fourier transformation, the tunneling density of states is thereby obtained

$$G^>(E) = \theta(E)2\pi\Gamma(m)\tau_c^m E^{m-1}. \qquad (4.11)$$

For $m = 1$, corresponding to the IQHE at $\nu = 1$, the tunneling density of states is a constant at zero energy (the Fermi energy). From (4.6) this gives an Ohmic I-V characteristic, with a tunneling conductance, I/V, proportional to t^2. This is the expected result, consistent with Landauer transport theory (4.1). However, for $m > 1$ the tunneling density of states *vanishes* at zero energy, giving rise to a non Ohmic I-V characteristic[17,28]:

$$I \propto t^2|V|^{2m-2}V. \qquad (4.12)$$

The linear conductance is strictly zero! At finite temperatures the density of states is sampled at $E \approx kT$, and a non-zero (linear) conductance is expected. Generalizing Fermi's Golden rule to $T \neq 0$ gives the expected result for the conductance[17,28]:

$$G \propto t^2 T^{2m-2}. \quad (4.13)$$

For $\nu = 1/3$ the predicted conductance vanishes with a large power of temperature, $G \propto T^4$.

In striking contrast to the IQHE, the FQHE point contact conductance *vanishes* identically at zero temperature. What is the physical origin of this difference? At the edge of an IQHE fluid the electrons behave at low energies as if they were not interacting - the edge state is a Fermi liquid. Thus an electron can be added or removed from the edge without appreciably disturbing the other electrons. In contrast, the electrons at the edge of a Laughlin FQHE fluid are in a highly correlated state. Indeed, after removal of an electron from the edge, the remaining electrons are *not* left in the groundstate of the edge with one less electron. Rather, the resulting state contains a "shakeup" spectrum of many low energy edge excitations, and is orthogonal to the groundstate. It thus follows that the many body matrix element for tunneling in (4.5) vanishes. This is a nice example of a general phenomenon known as the *orthogonality catastrophe*[41], which arises in such contexts as the Kondo problem and the X ray edge problem.

This orthogonality catastrophe is directly accessible to measurement. Fig. 9 shows data[29] for the conductance as a function of temperature through a point contact in an IQHE fluid at $\nu = 1$ and a FQHE fluid at $\nu = 1/3$. The difference in behavior is striking. For $\nu = 1$ the conductance approaches a constant at low temperatures, whereas for $\nu = 1/3$ the conductance continues to decrease upon cooling. Moreover, the low temperature behavior for $\nu = 1/3$ is consistent with the T^4 dependence predicted in (4.13). In our view, this data provides the first compelling experimental evidence for the Luttinger liquid, a phase discussed theoretically over 30 years earlier.

It is instructive to re-cast the result (4.13) in the language of the renormalization group (RG)[28]. Specifically, the vanishing conductance in the FQHE indicates that the tunneling perturbation, t, is *irrelevant.* As shown in Appendix A, the lowest order RG flow equation can be obtained from the scaling dimension, Δ, of the tunneling operator $e^{im(\phi_L - \phi_R)}$. From the power law behavior of the Greens function in (4.10) we may deduce that the scaling dimension of $e^{im\phi_{L,R}}$ is $m/2$. It then follows that $\Delta = m$. The RG flow equation is then simply

$$\frac{dt}{d\ell} = (1-m)t. \quad (4.14)$$

For FQHE states with $m > 1$, t is irrelevant as expected. The perturbative results (4.12) and (4.13) can be obtained by integrating this RG flow equation until the cutoff is of order kT (or eV), giving $t_{eff} \sim t^{m-1}$ and $G \sim t_{eff}^2$.

2. Weak Backscattering Limit

Having established that the conductance of a FQHE point contact vanishes at $T = 0$ for weak tunneling, we now turn to the opposite limit in which the point contact is almost completely open. Consider then a bulk quantum Hall fluid in which the top and bottom edges are weakly coupled together, as depicted in Fig. 8b. As before, the low energy physics is at the edges, and can be described by the action,

$$S = S_T^0 + S_B^0 + S_{\rm tun.}. \quad (4.15)$$

Here S_T^0 and S_B^0 describe the top and bottom edge modes, respectively, and are given by the chiral boson action (4.3). In contrast to the weak tunneling limit, the charge which tunnels between the two edges is now tunneling through the quantum Hall fluid. It is therefore possible that a single Laughlin quasiparticle, with fractional charge e/m, could tunnel between the edges. This process can be described by a term of the form,

$$S_{\rm tun.} = \int d\tau v e^{i(\phi_T - \phi_B)}, \quad (4.16)$$

where v is the tunneling amplitude. In addition, higher order processes involving tunneling of multiple quasiparticles (or electrons) are also possible. However, as shown below, such processes are less "relevant" at low energies and temperatures.

Consider now applying a voltage V between the source and drain electrodes. In the absence of any coupling, the top and bottom edges would then be in equilibrium at chemical potentials differing by the voltage V. This results in the flow of a net edge current, $I = (1/m)(e^2/h)V$. Quasiparticle tunneling between the top and bottom edges backscatters charge, and will tend to reduce this current. The reduction may be computed perturbatively in v, also using the Golden Rule. In fact, the backscattering current will be given by the Golden rule expression (4.5), with two differences. First, the charge e in (4.5) must be replaced by the quasiparticle charge, $e^* = e/m$. Second, the electron tunneling operator (4.4) must be replaced by the quasiparticle tunneling term in (4.16). This difference is crucial, replacing the electron tunneling DOS (4.11) with the density of states for the addition of a *quasiparticle*. This follows from the quasiparticle Green's function

$$G_{\rm q.p.}(\tau) = \langle T_\tau [e^{i(\phi(\tau) - \phi(0))}] \rangle \quad (4.17)$$

$$= \left(\frac{\tau_c}{|\tau| + \tau_c} \right)^{1/m}, \quad (4.18)$$

which has the same form as the electron Greens function (4.10) with $m \to 1/m$. The backscattering current can thus be obtained from (4.12) by replacing m with $1/m$, giving at zero temperature[17,28]

$$I_{\rm back} \propto v^2 |V|^{2/m-2} V. \quad (4.19)$$

Likewise, at temperature T, the backscattering contribution to the (linear) conductance is given by

$$G - \frac{1}{m}\frac{e^2}{h} \propto -v^2 T^{2/m-2}. \tag{4.20}$$

Once again, for the IQHE with $m = 1$, a temperature independent correction to the two terminal conductance is obtained, as expected from Landauer transport theory. However, for the FQHE the perturbation theory (4.20) is divergent at low temperatures. The quasiparticle tunneling rate *grows* at low energies, in contrast to electron tunneling. Rather than an orthogonality catastrophe, quasiparticle tunneling causes quite the opposite - an "overlap catastrophe".

The divergent perturbation theory indicates that the quasiparticle tunneling operator is a *relevant* perturbation. This may be seen directly by noting from (4.18) that the scaling dimension of the quasiparticle creation operator $e^{i\phi}$ is equal to $1/2m$. The leading order RG flow equation for the quasiparticle tunneling amplitude is then simply

$$\frac{dv}{d\ell} = (1 - \frac{1}{m})v. \tag{4.21}$$

For the FQHE, v grows upon scaling to lower energies, flowing out of the perturbative regime where (4.21) is valid. The behavior in this limit will be discussed in the next section.

It is instructive to consider, in addition, backscattering processes involving multiple quasiparticles. The operator which tunnels n-quasiparticles is of the form: $v_n e^{in(\phi_T - \phi_B)}$. It is straightforward to show that the leading order RG flow equation for v_n is

$$\frac{dv_n}{d\ell} = (1 - \frac{n^2}{m})v_n. \tag{4.22}$$

Notice that for $\nu = 1/3$, ($m = 3$), the single quasiparticle backscattering process is the only relevant perturbation. In contrast, for $m = 5, 7, ...$ more than one operator is relevant. In all cases, however, the single quasiparticle term is the most relevant.

3. Crossover between the two limits

The preceding results can now be pieced together to form a global picture of the behavior of a point contact in the FQHE. The above perturbative results describe the stability of two renormalization group fixed points. The "perfectly insulating" fixed point, with zero electron tunneling $t = 0$, is stable, whereas the "perfectly conducting" fixed point, with zero quasiparticle tunneling $v = 0$, is unstable. Provided these are the only two fixed points, it follows that the RG flows out of the conducting fixed point eventually make their way to the insulating fixed point. This is a very striking conclusion, since it implies that an arbitrarily weak quasiparticle backscattering amplitude v will cause the conductance to vanish completely at zero temperature. Of course, for v very small, very low temperatures would be necessary to see this. In this scenario, the conductance as a function of temperature will behave as shown in Fig. 10. At high temperatures, the system does not have "time" to flow out of the perturbative regime, so the conductance is given by $G \approx (1/m)(e^2/h) - v^2 T^{2/m-2}$. As the temperature is lowered below a scale $T^* \propto v^{m/(m-1)}$, perturbation theory breaks down. Eventually, the system crosses over into a low temperature regime in which the conductance vanishes as T^{2m-2}.

The validity of this scenario rests on the assumption that no other fixed points intervene. This assumption has been verified both by quantum Monte Carlo simulations[27], and more recently by exact non-perturbative methods based on the thermodynamic Bethe ansatz[31].

As seen from (4.22), for $\nu = 1/3$ the single quasiparticle backscattering operator, with amplitude $v = v_1$, is the *only* relevant perturbation about the conducting fixed point. All higher order processes are irrelevant, and hence less important at low temperatures. Indeed, for small v and T, the conductance will depend on these parameters only in the combination $v/T^{2/3}$. In this limit the conductance can be expressed in terms of a *universal crossover scaling function*[27,30,42]

$$\lim_{v,T\to 0} G(v,T) = \frac{1}{3}\frac{e^2}{h}\tilde{G}(cv/T^{2/3}), \tag{4.23}$$

where c is a non universal dimensionful constant. The limiting behavior of the scaling function, $\tilde{G}(X)$, may be deduced from the perturbative limits. For small argument X, the perturbation theory result (4.20) implies

$$\tilde{G}(X) = 1 - X^2. \tag{4.24}$$

For large argument, corresponding to the limit $T \to 0$, the scaling function must match on to the low temperature regime (4.13), which gives a T^4 dependence for $nu = 1/3$. This implies that for $X \to \infty$,

$$\tilde{G}(X) \propto X^{-6}. \tag{4.25}$$

The exact scaling function, recently computed by Fendley et. al.[31], indeed reduces to (4.23) and (4.24) for small and large argument, respectively. The universal crossover scaling function $\tilde{G}$ is of particular interest because it determines the experimentally accessible lineshape for resonant tunneling, as we now describe.

B. Resonant Tunneling

We now consider the phenomena of resonant tunneling through a point contact in the FQHE. Resonances in the conductance are expected when the energy of the incident edge mode coincides with a localized state in the vicinity of the point contact. As a point of reference, we first review resonant tunneling theory for a non-interacting electron gas, which should be applicable to a point contact in the IQHE. As the chemical potential μ of the incident edge mode sweeps through the energy of the localized state, ϵ_0, the conductance will exhibit a peak described by,

$$G = \frac{e^2}{h}\int d\epsilon f'(\epsilon-\mu)\frac{\Gamma_L\Gamma_R}{(\epsilon-\epsilon_0)^2+\Gamma^2}. \tag{4.26}$$

Here Γ_L and Γ_R are tunneling rates from the resonant (localized) state to the left and right leads and $\Gamma = (\Gamma_L + \Gamma_R)/2$. The Fermi function is denoted $f(\epsilon)$. At high temperatures, $T > \Gamma$, the resonance has an amplitude Γ/T and a width T. At low temperatures, the lineshape is Lorentzian, with a temperature independent width. Moreover, when the left and right barriers are identical, the on-resonance transmission at zero temperature is perfect, $G = e^2/h$.

How is this modified for tunneling through a FQHE point contact? Since arbitrarily weak quasiparticle backscattering causes the zero temperature conductance through the point contact to vanish, one might expect that resonances are simply not present at $T = 0$. As we now show, this is not the case. Rather, perfect resonances are possible, but in striking contrast to (4.26) for non- interacting electrons, they become infinitely sharp in the zero temperature limit. As before, it is useful to consider perturbatively two limits, the weak tunneling limit in Section 1 below, and then in Section 2 the opposite limit of weak backscattering.

1. *Weak Tunneling limit*

Consider then tunneling through a localized state separating two FQHE fluids. Focusing once again on the FQHE edge modes, we consider the model,

$$S = S_L^0 + S_R^0 + S_{\text{res.}} + S_{\text{tun.}} \tag{4.27}$$

where S_L^0 and S_R^0, given in (4.3), describe the edge modes in the two FQHE fluids, and

$$S_{\text{res.}} = \int d\tau \epsilon_0 d^\dagger d \tag{4.28}$$

describes the localized state with energy ϵ_0. The edge modes are coupled to the localized state via a tunneling term,

$$S_{\text{tun.}} = \int d\tau t(e^{i\phi_L} + e^{i\phi_R})d + h.c, \tag{4.29}$$

with the tunneling amplitude, t, taken to be the same for left and right edge modes. Once again, the boson fields $\phi_{L/R}$ are evaluated at $x = 0$.

Consider first computing the rate, Γ, for an electron to tunnel from the localized state into the edge modes, perturbatively in t. From Fermi's golden rule, this will depend on the density of states for tunneling into the edge, which is given in (4.11). We thus find

$$\Gamma = t^2 G^{>}(\epsilon_0 - \mu). \tag{4.30}$$

At finite temperatures, and $\mu \approx \epsilon_0$, we thus have

$$\Gamma \propto t^2 T^{m-1}. \tag{4.31}$$

Once again at zero temperature there is an orthogonality catastrophe which prevents tunneling to FQHE edge states, $m > 1$. However, it is only half as severe as that for tunneling between two edge modes (4.13), since only a single mode is being disturbed. For FQHE states ($m \geq 3$), (4.31) implies that $\Gamma << T$ at low temperatures. In Ref. 43, 44 it was argued that in this limit the conductance is well approximated by the form (4.26) with a temperature dependent tunneling rate Γ in (4.31). This implies resonances with a width varying as T and a height $\Gamma(T)/T$, which gives an on-resonance conductance varying as:

$$G_{\text{res.}} \propto t^2 T^{m-2}. \tag{4.32}$$

Thus, in the limit of small tunneling t, resonances are indeed suppressed at zero temperature. However, at finite temperatures, peak heights are predicted to vanish more slowly (as T for $\nu = 1/3$) than the tails (T^4 for $\nu = 1/3$).

What happens when the tunneling, t, to the localized state is increased? In Ref. 42 a renormalization group calculation was performed to higher order in t, which revealed that the exponent in (4.32) is renormalized when t is finite. Specifically, to $O(t^2)$ the following RG flow equations were obtained,

$$\frac{dt}{d\ell} = (1 - \frac{1}{2}(m-\alpha))t \tag{4.33}$$

$$\frac{d\alpha}{d\ell} = 8\tau_c^2 t^2 (1 - \frac{2\alpha}{m}). \tag{4.34}$$

The RG flow is shown in Fig. 11. Initially, $\alpha = 0$, however, when t is finite it becomes positive. For small t, the RG flows to a fixed line with $t = 0$, and $\alpha = \alpha^*$. The on-resonance conductance then decays with a modified exponent,

$$G_{\text{res.}} \propto t^2 T^{m-2-\alpha^*}. \tag{4.35}$$

For $\nu = 1/3$ ($m = 3$) when t is larger than a critical value t_c, for which $\alpha_c^* = m - 2$, the flows cross a Kosterlitz-Thouless like separatrix, and scale towards large tunneling t (see Fig. 11). In this case, it is extremely plausible that the flows take the system all the way to the perfectly conducting fixed point, as argued in Ref. 42. This certainly happens for the IQHE $m = 1$, since (4.26) shows a perfect on-resonance conductance when $\Gamma_L = \Gamma_R$. (In this case the renormalization of α is inconsequential, however, since from (4.33) t is always relevant.) For the FQHE the equality of the left and right tunneling amplitudes assumed in (4.29) is critical. Indeed, when they are unequal, the RG flows are modified, and the system crosses over to off-resonance behavior (4.13). For $m \geq 5$ the RG flows reveal that the tunneling amplitude always scales to zero. Thus for $\nu \leq 1/5$ perfect resonances are not readily attainable.

In summary, we conclude that for a localized state coupled symmetrically to two $\nu = 1/3$ FQHE fluids, robust resonances are indeed possible. Since the conductance on resonance is expected to be large, the above perturbative analysis in t cannot be used to calculate the resonance lineshape. The behavior near the resonance peak, however, can be obtained easily in the opposite limit of weak backscattering, as we now describe.

2. Weak Backscattering: Theory of the Perfect Resonance

Resonant tunneling is not normally studied for weak backscattering, since in this limit the transmission is large even off resonance, which tends to obscure the resonance. However, for FQHE states, the off resonance conductance vanishes at zero temperature, leaving an unobscured resonance peak.

Consider a point contact which has two nearby parallel tunneling paths for the backscattering of quasiparticles, as depicted in Fig. 12. These tunneling paths may be due to a random impurity potential or an intentionally created quantum dot. By varying the gate voltage and magnetic field it should be possible to achieve a destructive interference,

$$v_{\rm eff} = v_L + v_R = 0 \tag{4.36}$$

which shuts off the inter-edge quasiparticle tunneling. This is the condition for a resonance. Provided all higher order tunneling processes are irrelevant, there will be perfect transmission on-resonance at $T = 0$, with a conductance $G = \nu e^2/h$. Upon tuning away from the resonance, so $v_{\rm eff} \neq 0$, the conductance will vanish at zero temperature, as shown in Section A above. Thus, at zero temperature, there will be an *infinitely sharp perfect resonance*[30].

How easy is it to achieve such a perfect resonance? The criterion is that the renormalized value of $v_{\rm eff}$ and all other relevant v's equal zero. In general, the net quasiparticle backscattering amplitude $v_{\rm eff}$ is a complex number, so that the resonance condition requires the simultaneous tuning of two parameters. If barriers (or quasiparticle tunneling paths) are symmetric, however, then $v_{\rm eff}$ may be chosen real, so that only a single parameter, such as a gate voltage, need be tuned. For $\nu = 1/3$, all higher order backscattering processes are indeed irrelevant, so that tuning $v_{\rm eff} = 0$ is sufficient to achieve resonance. For $\nu = 1/5$, however, the parameter v_2 in (4.22) is also relevant, so a perfect resonance requiring the tuning of 4 parameters. The situation gets even worse for $m > 5$. For this reason, we focus on resonances for $\nu = 1/3$, which should be the easiest to find.

Consider tuning through such a perfect resonance by varying a parameter, such as the gate voltage. It is convenient to denote by δ the "distance" from the peak position in the control parameter. Close enough to the resonance one has $v_{\rm eff} \propto \delta$. For very small δ the RG flows will thus pass very near to the perfectly conducting fixed point, since all of the other irrelevant operators will scale to zero before $v_{\rm eff}$ has time to grow large. Eventually, $v_{\rm eff}$ does grow large and the flows crossover to the insulating fixed point, as depicted in Fig. 13. Temperature serves as a cutoff to the RG flows, as usual. This reasoning reveals that for both δ and temperature small, the conductance will depend only on the universal crossover trajectory which joins the two fixed points. The uniqueness of the RG trajectory implies that the conductance will be described by a universal crossover scaling function. Thus, for small T and δ, the resonance lineshape is given by a universal scaling function,

$$G(T,\delta) = \frac{1}{3}\frac{e^2}{h}\tilde{G}(\delta/T^{2/3}), \tag{4.37}$$

where $\tilde{G}$ is the same scaling function introduced in (4.23).

The scaling form (4.37) shows that the resonance width scales as $T^{2/3}$, at low temperatures. Moreover, rescaled data from different temperatures should collapse onto the same universal curve. As seen from (4.25), the resonance lineshape is predicted to be *non-Lorentzian*, with a tail falling off as δ^{-6}.

Fig. 14 shows a scaling plot of one of the resonances for $\nu = 1/3$ in Fig. 7. from the data of Webb et. al.. The widths of the resonances at several different temperatures have been rescaled by $T^{2/3}$, as suggested by (4.37). Since the peak heights were also weakly temperature dependent (and roughly one third of the quantized value $(1/3)e^2/h$) the amplitudes have also been normalized to have unit height at the peak. The temperature scaling of the peak widths is indeed very well fit by $T^{2/3}$. Also shown in Fig. 14 are quantum Monte Carlo data and an exact computation from Bethe Ansatz for the universal scaling function in (4.37). The agreement is striking. Although the experimental lineshape does drop somewhat faster in the tails, the shape is distinctly non Lorentzian with a tail decaying with a power close to that predicted by theory. It should be emphasized that the experimental data does not represent a "perfect resonance", since the peak amplitude is dropping (slowly) upon cooling, rather than approaching the quantized value, $(1/3)e^2/h$. By varying an additional parameter besides the gate voltage (such as the magnetic field) though, it should be possible to find a perfect resonance for $\nu = 1/3$.

C. Generalization to Hierarchical States

The above theory of tunneling through a point contact for FQHE fluids at filling $\nu = 1/m$, can be generalized to hierarchical FQHE states[25]. The additional complication is that the hierarchical states have composite edges, with multiple branches, as described in detail in Section II. Moreover, for those states with edge branches moving in both directions, such as $\nu = 2/3$, the conductance is non-universal unless edge impurity scattering is present. Likewise, it can be shown that without impurity scattering, the conductance through a point contact in a $\nu = 2/3$ fluid varies as

$$G \sim T^{\alpha} \tag{4.38}$$

with a *non-universal* exponent α. However, impurity scattering drives an edge phase transition, as shown in Section III, and the system flows to a fixed point which exhibits an appropriately quantized conductance. At this fixed point, the exponent α is likewise universal. Thus, measuring α, gives critical information about the low energy properties of the composite edge.

Generally, the exponent α can be determined from the density of states to tunnel an electron into the composite edge. Specifically, one must consider all tunneling operators of the form (2.45), which create an edge excitation with charge e, as determined from (2.34). For filling factors $\nu = n/(np+1)$, with integer n and even integer p, the edge fixed point with impurity scattering is known (see Section IIIc) so that the tunneling DOS can be readily computed, by generalizing (4.9). For $\nu = 2/3$ one finds

$$\alpha = (2/\nu - 2) + 1. \tag{4.39}$$

This exponent *depends* on the presence of the neutral mode at the $\nu = 2/3$ edge! The electron edge creation operator is a combination of the charge and neutral modes, so that tunneling an electron into the edge also "shakes up" the neutral mode. Indeed, the second contribution in (4.39) is due to shakeup of the neutral mode. The charge mode gives the first term, which has the same form as (4.14) for filling $\nu = 1/m$. Thus, an observation of $\alpha = 2$ for a $\nu = 2/3$ point contact, would constitute a measurement of the neutral mode!

More generally for $\nu = n/np + 1$ it can be shown that

$$\alpha = \begin{cases} 2p & \text{for } p \geq 0 \\ 2|p| - (4/n) & \text{for } p < 0. \end{cases} \tag{4.40}$$

Notice that α depends on the sign of p, which determines the direction of propagation of the neutral modes relative to the charge mode. For the $p = -2$ sequence, the predicted exponents are displayed in table 1. The exponents approach $\alpha = 4$ as ν approaches $1/2$.

For filling $\nu = 4/5$ the conductance at $T = 0$ was argued in Section IIIc to be non-universal even in the presence of edge impurity scattering, although at $T \neq 0$ a universal quantized conductance is restored for samples larger than the edge equilibration length (3.29). In contrast, the exponent α is predicted to be *non-universal* even for samples much longer than the edge equilibration length.

For a point contact which is only very weakly pinched off, the conductance can be computed perturbatively for small backscattering. The backscattering will reduce the conductance from its quantized value, as in the single channel case (4.20). Generally, for $\nu = n/(np + 1)$ we find a temperature dependent suppression given by,

$$G(T) = |\nu| \frac{e^2}{h} - v^2 T^{2(|\nu|-1)}, \tag{4.41}$$

where v is the amplitude of the most relevant backscattering operator of the general form (2.45). If there were no other relevant backscattering operators, as for $\nu = 1/3$, this would imply that resonances narrow with temperature as $T^{(1-|\nu|)}$ - the generalization of (4.37). However, for hierarchical states there will generically be several relevant backscattering processes, so that resonances will not be very robust, and tend to vanish at very low temperatures. Nevertheless, one expects there should be a range of temperature over which the resonance width narrows as $T^{(1-|\nu|)}$.

D. Shot Noise

In addition to measuring the conductance of a point contact, it is also possible to measure time dependent fluctuations, or noise, in the transmitted current. Most interesting is the non-equilibrium noise present at finite bias voltage, rather than the equilibrium Nyquist noise. At frequencies comparable to the bias voltage, there may be Josephson type oscillations, as discussed by Chamon et. al.[45]. At low frequencies one expects shot noise, arising from the discreteness of the electron. As we briefly describe, shot noise might enable a rather direct measurement of the fractional charge of the Laughlin quasiparticle[46].

Consider first a very high resistance point contact in a QHE fluid. In the presence of a bias voltage electrons will occasionally tunnel. This will give rise to temporal fluctuations in the current. At low enough currents, successive tunneling events will be uncorrelated. Assuming Poisson statistics for the tunneling events, the low frequency current noise will be given by the classic expression:

$$\langle |\delta I(\omega)|^2 \rangle_{\omega \to 0} = e \langle I \rangle. \tag{4.42}$$

Note that the amplitude of the low frequency noise depends on the absolute magnitude of the electric charge, e.

With increasing current, correlations between tunneling events are expected, and the above expression must break down. Recently Lesovik[47–49] has analyzed shot noise for a 1d non-interacting electron gas, which is relevant to a $\nu = 1$ point contact. For a single barrier with transmission probability $|t|^2$, he finds

$$\langle|\delta I(\omega)|^2\rangle_{\omega\to 0} = \frac{e^2}{h}|t|^2(1-|t|^2)eV, \tag{4.43}$$

where V is the bias voltage. In the limit $|t|^2 \to 0$, this reduces to the classic expression (4.42). But when $|t|^2 \to 1$, the noise is greatly suppressed. Indeed, in the absence of any backscattering ($|t|^2 = 1$) the noise vanishes altogether. For $1 - |t|^2$ small (4.43) may be re-written as

$$\langle|\delta I(\omega)|^2\rangle_{\omega\to 0} \approx e\langle I_{\rm back}\rangle = e(\frac{e^2}{h}V - \langle I\rangle). \tag{4.44}$$

The noise arises from the discrete, uncorrelated *backscattering* of electrons at the point contact.

How are these results modified for a point contact in the FQHE? In Ref. 46 we developed a detailed theory of non-equilibrium shot noise at FQHE point contacts for filling $\nu = 1/m$. The results may be understood very simply. In the weak tunneling limit the transport is dominated by the discrete tunneling of electrons through the point contact. As for non-interacting electrons, the tunneling events satisfy Poisson statistics and the noise is given by (4.42). In the opposite limit of weak backscattering, however, there are qualitative differences, because the dominant backscattering processes in the FQHE are fractionally charged quasiparticles. When the conductance is just slightly less than $\nu e^2/h$, (attained by adjusting the gate on the point contact) these backscattering processes are infrequent, and should be uncorrelated. Indeed, in this limit we find a low frequency noise given by

$$\langle|\delta I(\omega)|^2\rangle_{\omega\to 0} \approx e^*\langle I_{\rm back}\rangle = e^*(\nu\frac{e^2}{h}V - \langle I\rangle), \tag{4.45}$$

with e^* the quasiparticle charge: $e^* = e/m$. This form is identical to the non-interacting result (4.44), except with the electron charge replaced by the quasiparticle charge. A measurement of the current noise and mean current, $\langle I\rangle$, in this regime should enable a direct measure of the quasiparticle charge.

V. CONCLUSION

In this article we have presented a theory of edge state transport in the fractional quantum Hall effect based on the chiral Luttinger liquid model. For the Laughlin states at filling $\nu = 1/m$ with odd m, this model provides a very simple description of the low energy edge excitations, which consist of a single propagating mode corresponding to charge density fluctuations. This provides a simple framework for understanding the quantization of the Hall conductance in an edge state picture, analogous to the Landauer-Buttiker theory for the integer quantum Hall effect.

In addition, this theory makes specific, experimentally testable predictions for the behavior of tunneling and resonant tunneling through a point contact in a Hall fluid. The behavior for a FQHE fluid is predicted to be qualitatively different than that in the integer effect. Specifically, for $\nu = 1/3$ the conductance through a point contact is predicted to vanish at low temperatures as T^4, in contrast to the temperature independent result expected for $\nu = 1$. Moreover, resonances in the tunneling between two $\nu = 1/3$ states are predicted to have a temperature dependent line width, which vanishes as $T^{2/3}$ at low temperatures. The shape of the resonances are universal and described by a scaling function which has been computed exactly. These predictions agree qualitatively - if not quantitatively - with recent measurements of edge state transport through a point contact at $\nu = 1/3$[29].

It is worth emphasizing that a point contact in a QHE fluid provides a simple and experimentally accessible example of a broad class of so-called "quantum impurity problems". Quantum impurity problems[50] consist of an "impurity" which is coupled to an extensive set of low energy degrees of freedom. The classic example is in the Kondo effect[51], where an impurity spin is coupled to the particle-hole excitations of a metallic host. In the quantum Hall effect, the point contact is an impurity, coupled to the low energy edge excitations. The powerful methods of boundary conformal field theory[50] and "boundary integrability" are ideal for analyzing this class of problems.

In contrast to the Laughlin sequence, $\nu = 1/m$, the edge excitations of hierarchical quantum Hall states cannot be described by a single mode. Rather, multiple Luttinger liquid edge modes are expected, which in general can propagate in different directions. This can lead to a breakdown of conductance quantization due to a lack of equilibration between opposite moving modes. It is thus essential to incorporate edge impurity scattering, which has a profound effect on the low energy edge state structure. Specifically, for a broad class of hierarchical quantum Hall states, $\nu = n/(np + 1)$ with integer n and even integer p, impurity scattering is perturbatively relevant, and drives the edge to a new disorder dominated low energy fixed point, where quantization is restored. At this fixed point the charge is carried in a single mode. The remaining $n - 1$ neutral modes propagate at a different velocity and are related by an exact $SU(n)$ symmetry. In the specific case $\nu = 2/3$, there is a single neutral mode, which propagates in the direction opposite to the charge mode.

Despite carrying no charge, the upstream propagating neutral mode can be detected in at least two ways. The first, which is less direct, involves tunneling through a constricted point contact in a $\nu = 2/3$ Hall fluid. For filling $\nu = 1/m$, where there is only a single edge mode, the point contact conductance is predicted to vanish with temperature as $G(T) \sim T^{2/\nu-2}$. For $\nu = 2/3$ one might therefore expect a power law, $G(T) \sim T$. However, the presence of the neutral mode at the $\nu = 2/3$ edge increases this power by one, giving the prediction $G(T) \sim T^2$.

Time domain transport experiments at filling $\nu = 2/3$, might enable a much more direct measurement of the neutral mode. Imagine two leads coupled via tunnel junctions to the opposite sides of a large Hall droplet at filling $\nu = 2/3$. A short current pulse incident in one lead, upon tunneling into the droplet edge, would excite both the charge and neutral edge modes. These excitations, after propagating in opposite directions and with different speeds along the droplet edge would, upon arrival at the far tunnel junction, excite *two* current pulses into the outgoing lead. By tailoring the placement of the leads, a measurement of the direction of propagation and decay length of the neutral mode should also be possible.

The disorder dominated fixed point which describes the edge structure at filling $\nu = n/(np+1)$ has higher symmetry - an exact U(1)xSU(n) symmetry - than the underlying Hamiltonian. This is reminiscent of Fermi liquid theory, where the attractive zero temperature fixed point also has higher symmetry than the underlying Hamiltonian. In addition to conserved electric charge, the (T=0) Fermi liquid fixed point has an infinity of conserved charges (and hence an infinity of U(1) symmetries) associated with each point on the Fermi surface. This is the symmetry responsible for the quasiparticle excitations. At finite temperatures this symmetry is broken, leading to a finite scattering lifetime for the quasiparticles, proportional to T^{-2}. Since total electric charge is always conserved, propagating zero sound in a Fermi-liquid does not decay even at $T \neq 0$. At the FQHE edge, it is the SU(n) symmetry which is responsible for the existence of the neutral modes. But, as in Fermi liquid theory, this symmetry is only exact *at* $T = 0$, so that the neutral edge excitations are expected to decay at finite temperatures. It is amusing that the predicted scattering rate for the neutral modes vanishes with the same power of temperature - T^2 - as for quasiparticles in a Fermi liquid.

In conclusion, edge states of fractional Hall fluids provide an ideal arena for the study of correlations in low dimensional quantum transport. The remarkable richness of the edge state structure in both the Laughlin states and the hierarchical quantum Hall states is directly accessible to experimental study. We hope that this overview will stimulate further experimental and theoretical work in this exciting area.

ACKNOWLEDGMENTS

We gratefully acknowledge our collaborators in this work, Steve Girvin, Kyungsun Moon, Joe Polchinski and Hangmo Yi. We also thank A.H. MacDonald, N. Read, M. Stone, R.A. Webb, X.G. Wen and A. Zee for fruitful discussions. M.P.A.F. has been supported by the National Science Foundation under grants No. PHY89-04035 and No. DMR-9400142.

APPENDIX A: RENORMALIZATION GROUP ANALYSIS

The renormalization group (RG) provides a powerful framework for understanding the global behavior of the models in this article and for piecing together results obtained in perturbative limits. Here we outline the procedure for deriving the lowest order RG flow equations which are referred to in the text.

Consider first a point contact in a $\nu = 1/m$ fluid in the limit of weak backscattering, discussed in Section IVA. The action in (4.15)-(4.16) may be written

$$S = S_0 + v \int d\tau e^{i\phi(\tau)} + c.c., \tag{A1}$$

where $S_0 = S_T^0 + S_B^0$ is the quadratic edge action given in (4.3) and $\phi(\tau) = \phi_T(x=0,\tau) - \phi_B(x=0,\tau)$. The perturbation v acts at a single space point, $x = 0$.

The RG of two steps: (*i*) Integrate out degrees of freedom $\phi_{T/B}(k,\omega)$ which lie in a momentum shell $\Lambda/b < k < \Lambda$. First split the field into "slow" and "fast" modes, below and inside the momentum shell, respectively: $\phi = \phi_s + \phi_f$. To lowest order in v one must average over the fast modes:

$$\langle e^{i\phi} \rangle_f = e^{i\phi_s} \langle e^{i\phi_f} \rangle_f \tag{A2}$$

$$= b^{-\Delta} e^{i\phi_s}. \tag{A3}$$

Here Δ is the scaling dimension of the operator $e^{i\phi}$. The scaling dimension is most easily deduced from the two point correlation function,

$$\langle e^{i\phi(\tau)} e^{-i\phi(0)} \rangle \propto |\tau|^{-2\Delta}. \tag{A4}$$

(*ii*) The RG transformation is completed by rescaling space and time, $\tau' = \tau/b$, $x' = x/b$. The resulting action is then equivalent to the original one with v replaced by $v' = v b^{1-\Delta}$. Upon setting $b = e^{\ell}$, one thereby obtains the leading order differential RG flow equation,

$$\frac{dv}{d\ell} = (1-\Delta) v. \tag{A5}$$

Consider now a spatially random perturbation, as in section III,

$$S = S_0 + \int dx d\tau \xi(x) e^{i\phi(x,\tau)} + c.c. \tag{A6}$$

where $\xi(x)$ is a Gaussian random variable satisfying $[\xi^*(x)\xi(x')]_{ens} = W\delta(x-x')$. The square brackets denote an ensemble average over the quenched disorder. Our analysis follows closely that of Giamarchi and Schultz[52], who studied the effects of randomness on a (non chiral) Luttinger liquid. To *lowest order* in W, ensemble averaged quantities may be computed from the ensemble average of the partition function. (At higher order the introduction of replicas would be useful.) Performing the average over ξ, give the effective action

$$S_{\rm eff} = S_0 - W \int dx d\tau d\tau' e^{i\phi(x,\tau)} e^{-i\phi(x,\tau')}. \qquad (A7)$$

The leading order RG flow equation for W may now be derived by applying steps (i) and (ii) to (A7). This gives

$$\frac{dW}{d\ell} = (3 - 2\Delta) W, \qquad (A8)$$

where again Δ is the scaling dimension of $e^{i\phi}$. The "2" arises from step (i) because $e^{i\phi}$ appears twice in (A7). The "3" arises from rescaling, step (ii), because there are 3 space/time integrals.

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FIG. 1. Schematic portrait of the edge of a quantum Hall state with two channels. The solid lines with arrows represent the edge states. The presence of random impurities, denoted by the small circles, allows for momentum non-conserving scattering between the different channels. When the channels move in the same direction (e.g. $\nu = 2$), as shown in (a), inter-channel scattering does not effect the net transmission of the edge. However, when the channels move in opposite directions, as in $\nu = 2/3$, depicted in (b), the back scattering of charge plays a crucial role.

FIG. 2. Schematic portrait of a point contact, in which the top and bottom edges of a Hall fluid are brought together by an electrostatically controlled gate (G), allowing for the tunneling of charge between the two edges. Here S and D denote source and drain, respectively.

FIG. 3. Dispersion of energy levels in a quantum Hall bar as a function of the one dimensional momentum k. Here μ is the Fermi energy at bulk filling $\nu = 1$.

FIG. 4. Schematic diagram of a two terminal conductance measurement for a $\nu = 1$ quantum Hall state. The shaded regions denote the reservoirs, which are assumed to be in equilibrium at different chemical potentials.

FIG. 5. Schematic diagram of a two terminal conductance measurement for a quantum Hall state such as $\nu = 2/3$ in which two edge channels move in opposite directions.

FIG. 6. Renormalization group flow diagram for a $\nu = 2/3$ random edge as a function of disorder strength W and the scaling dimension Δ of the tunneling operator. For $\Delta < 3/2$ all flows end up at the exactly soluble fixed line $\Delta = 1$. For $\Delta > 3/2$ there is a Kosterlitz-Thouless like separatrix separating the disorder dominated phase from a phase in which disorder is irrelevant.

FIG. 7. Two terminal conductance as a function of gate voltage of a GaAs quantum Hall point contact taken at 42mK. The two curves are taken at magnetic fields which correspond to $\nu = 1$ and $\nu = 1/3$ plateaus. Taken from Ref. 29.

FIG. 8. A quantum Hall point contact in the (a) weak tunneling limit and (b) the weak backscattering limit. The shaded regions represent the quantum Hall fluid with edge states depicted as lines with arrows. The dashed line represents a weak tunneling matrix element connecting the two edges.

FIG. 9. Conductance of a quantum Hall point contact as a function of temperature for (a) $\nu = 1$ and (b) $\nu = 1/3$ from Ref. 29.

FIG. 10. Schematic plot of the crossover from the weak backscattering limit to the weak tunneling limit as the temperature is lowered. At high temperatures, weak backscattering leads to a small correction to the quantized conductance. As the temperature is lowered below $T^* \propto v^{m/(m-1)}$ the system crosses over to the insulating limit.

FIG. 11. Renormalization group flow diagram describing the on resonance transmission. For weak tunneling, small t, the system flows to the fixed line with $t = 0$. The peak conductance then vanishes at low temperature as $T^{m-2-\alpha^*}$. As the coupling t is increased, the system crosses a Kosterlitz Thouless separatrix, and flows to the $v = 0$ fixed point described in section IVB2.

FIG. 12. A quantum Hall point contact with two parallel backscattering paths, v_1 and v_2. A perfect resonance occurs when the backscattering amplitudes destructively interfere.

FIG. 13. Schematic renormalization group flow diagram showing the universal trajectory connecting the perfectly conducting $v = 0$ fixed point to the insulating $t = 0$ fixed point. On resonance, the system flows into the unstable $v = 0$ fixed point. Slightly off resonance, the system flows past the $v = 0$ fixed point and flows along the universal trajectory to $t = 0$.

FIG. 14. Log-log scaling plot of the lineshape of resonances at different temperatures from Ref. 29. The x axis is rescaled by $T^{2/3}$. The crosses represent experimental data at temperatures between $40mK$ and $140mK$. The squares are the results of the Monte Carlo simulation, and the solid line is the exact solution from Ref. 31.