**Physics 150**
**Position, Velocity and Acceleration**

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Accurately describing the motion of objects is a daunting task. Y+F's Chapter 1 develops a conventional approach to *kinematics* which gives a description of motion that will be amenable to later analysis with Newton's Laws. One of Newton's deep insights was that even some very complex motions can be simplified by thinking not about the motion itself but instead about successive *changes* in the state of motion.

Let's first collect some vocabulary from Ch. 1 to do this. To identify the *position* of a point object in a three dimensional space (like a classroom or office) one needs three numbers which locate the object "left or right" ($x$), "forward or backward" ($y$) and "up or down" ($z$). These are collected into a vector-valued quantity called the position $\mathbf{r}$:

$$\mathbf{r} = (x, y, z) = x\,\hat{\mathbf{x}} + y\,\hat{\mathbf{y}} + \;z\,\hat{\mathbf{z}} \tag{1}$$

The trailing vectors are unit vectors (they are dimensionless and have length "1") pointing in three perpendicular directions. Chapter 1 reminds you that position vectors have simple rules for addition and subtraction that amount to the ordinary rules of arithmetic but applied to each of the components separately. Equation 1 can be regarded as giving the overall distance of the object from an observer as the magnitude $r = \sqrt{x^2 + y^2 + z^2}$ and if we divide by the magnitude $\mathbf{r}/r = \hat{\mathbf{r}}$ we get a three-component unit vector $\hat{\mathbf{r}}$ that points along the line of sight from the observer to the object.

If the object is moving, all three components can change as functions of time $t$: $\mathbf{r}(t) = (x(t), y(t), z(t))$. The *velocity* is the ratio of the change of position $\Delta\mathbf{r}$ to the change of the time $\Delta t$. There are two variants of this quantity. Let's suppose that in the time interval $t_1 \to t_2$ the position changes $\mathbf{r}_1 \to \mathbf{r}_2$. The *average velocity* is the ratio

$$\mathbf{v}_{\text{av}} = \frac{\mathbf{r}_2 - \mathbf{r}_1}{t_2 - t_1}$$

For example, if you walk from DRL to the Penn Bookstore in 8 minutes = 480 s you would walk north approx. 50 m on 33rd street ($y$) and then west approx −500 m on Walnut Street ($x$). We will ignore any change of elevation ($z$) (it's not zero but I don't actually know what it is). Then the average velocity is

$$\mathbf{v}_{\text{av}} = \left(\frac{\Delta x, \Delta y, \Delta z}{\Delta t}\right) = (-1.04, .104, 0)\ \text{m/s}$$

If one takes the averaging interval between $t_1$ and $t_2$ to be infinitesimally small, i.e. $t_2 = t_1 + \delta t$ then the ratio defines the *instantaneous* velocity, or more simply just the velocity

$$\mathbf{v}(t) = \lim_{\delta t \to 0} \frac{\mathbf{r}(t + \delta t) - \mathbf{r}(t)}{\delta t}$$

For example when you walk to the bookstore, as you leave DRL and head 50 m north on 33rd street for 30 s (say at a constant pace) your velocity *at each instant of time* is

$$\mathbf{v} = \left(0, \frac{dy}{dt}, 0\right) = (0, 1.67, 0)\ \mathrm{m/s}$$

Note that here only the $y$ component is nonzero here since you are only walking north. The instantaneous velocity $\mathbf{v}$ is the same at every instant of the straight-line walk at a constant pace along 33rd Street and the direction of $\mathbf{v}$ changes suddenly when you eventually turn west onto Walnut street. It is possible that the magnitude $v$ (called the speed) would also change when you turn the corner.

If we further consider only motion in one dimension (walking in a straight line with no turns, tossing a tennis ball vertically, etc) this has a useful geometric interpretation. Make a graph of $x$ (along the vertical axis called the ordinate) versus $t$ (horizontal axis: abscissa). The velocity $dx/dt$ is the slope this graph at each instant of time. Conversely the distance moved in a (not necessarily small) interval of time is the area under a graph of $v$ (ordinate) versus $t$ (abscissa). This is a signed quantity: if $v > 0$ the area between the $v \cdot t$ graph and the $t$ axis is regarded as positive ($x$ increasing) and if $v < 0$ negative (decreasing). The first couple of active learning exercises present motion in one dimension where this identification turns out to be an extraordinarily powerful way to avoid tedious algebraic calculations.

One can also think about the changes of $\mathbf{v}$ using the same approach. The ratio of the changes in $\mathbf{v}$ to the elapsed time $\Delta t$ is called the *acceleration* $\mathbf{a}$ which is also a vector-valued quantity.

$$\begin{aligned} \mathbf{a}_{\mathrm{av}} &= \frac{\mathbf{v}_2 - \mathbf{v}_1}{t_2 - t_1} \\ \mathbf{a} &= \lim_{\delta t \to 0} \frac{d\mathbf{v}}{dt} \end{aligned}$$

For one dimensional motion the acceleration is the slope of a graph of $v$ versus $t$. A graph of $v(t)$ is often the most insightful way to study a complicated kinematics problem. This is for at least two reasons: from $v(t)$ in one step you can (a) get to $x(t)$ (take areas) *or* (b) get to $a(t)$ (take slopes). The homeworks provide a couple of examples where the $v \cdot t$ graph avoids a lot of otherwise annoyingly complicated analysis. When you really understand a subject you can often replace the math by graphs. (Hawking famously said that he always tried to think in terms of geometry instead of equations.)

The acceleration $\mathbf{a}$ represents the "second" rate of change of the position. This corresponds to the *curvature* of a graph of position as a function of time. One could repeat this construction by inspecting even higher rates of rates, but thankfully Newton tells us to stop here because he thought that the causes of motion could be encoded in the accelerations.

One important application of this machinery is for the special case where the acceleration is constant. This applies for example to free fall motion which describes (approximately) the motion of a small mass close to the surface of the earth, in which case the acceleration $a = -g \approx -9.8\,\mathrm{m/s^2}$ along the vertical (i.e. perpendicular to the surface of the earth). **Warning:** *To avoid confusion you should always interpret $g$ as a positive quantity and display the minus sign (denoting downward) explicitly in your calculations.* Failure to do this is responsible for almost all preventable errors we see from students in elementary mechanics.

Constant acceleration is an important special case: it is one of the very few situations in classical kinematics that has an exact analytic solution. Usually we aren't so lucky. Here is the explicit solution for motion along a single coordinate $x$ with constant acceleration $a$.

$$\boxed{x(t) = x(0) + v(0)t + \frac{1}{2}at^2} \text{ (constant } a) \tag{2}$$

This solution contains three constants: $x(0), v(0)$ and $a$. However these play very different roles in the solution. The combination $\{x(0), v(0)\}$ tell you how the motion got started (namely, where it was and how fast it was moving at $t = 0$). On the other hand $a$ is the acceleration *during* the motion. **Warning:** *Eqn. 2 can only be used when the acceleration is constant during the interval.* Chapter 2 translates this fundamental result into several useful alternate forms. I tend to avoid them, but one of my favorites is the very useful sum rule that says that if the acceleration is constant then the average velocity over a time $t$ is the *arithmetic mean* of the initial and final velocities at the beginning and end of the interval.

$$v_{\mathrm{av}} = \frac{v(t) + v(0)}{2}$$

There are several ways to prove this, and perhaps the cleanest is just to make a graph of $v$ as a function of $t$ and interpret distance moved as an area (I'll leave you to fill in the details).